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Well-balanced positivity preserving cell-vertex central-upwind scheme for shallow water flows

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ABSTRACT

We develop a new second-order two-dimensional central-upwind scheme on cell-vertex grids for approximating solutions of the Saint-Venant system with source terms due to bottom topography. Central-upwind schemes are developed based on the information about the local speeds of wave propagation. Compared to the triangular central-upwind schemes, the proposed cell-vertex one has an advantage of using more cell interfaces which provide more information on the waves propagating in different directions. We propose a new piecewise linear approximation of the bottom topography and a novel non-oscillatory reconstruction in which the gradient of each variable is computed using a modified minmod-type method to ensure the stability of the scheme. A new technique is proposed for the correction of the water surface elevation which guarantees the positivity of the water depth. The well-balanced property of the proposed central-upwind scheme is ensured using a special discretization for the cell averages of the topography source terms. The proposed scheme is tested on a number of numerical examples, among which we consider steady-state solutions with almost dry areas and their perturbations and solutions with rapidly varying flows over discontinuous bottom topography. Our numerical experiments confirm stability, well-balanced, positivity preserving properties and second-order accuracy of the proposed method. This scheme can be applied to shallow water models when the bed topography is discontinuous and/or highly oscillatory, and on complicated domains where the use of unstructured grids is advantageous.

1. Introduction

This paper focuses on development of modern numerical methods for the two-dimensional (2D) Saint-Venant system of shallow water equations (SWEs):

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x + (huv)_y = -ghB_x, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{g}{2}h^2\right)_y = -ghB_y. \end{cases} \tag{1}$$

Here, h is the water depth, $(u, v)^T$ is the velocity field, the function $B(x, y)$ represents the bottom elevation, and g is the acceleration due to gravity.

Many upwind (see, e.g., [1,2,5,9,10,18,22,31,33,37]) and central (see, e.g., [6,8,15,23,28,40,41,45]) schemes for the shallow water system (1), which is a hyperbolic system of conservation (if $B_x \equiv B_y \equiv 0$) or balance (if B is not a constant) laws, have been proposed in the past two decades. Roughly speaking, the main difference between upwind and central schemes is that upwind schemes use characteristic information and utilize (approximate) Riemann problem solvers to determine nonlinear wave propagation, while central schemes are based on averaging over the waves without using their detailed structures.

Riemann-problem-solver-free central schemes have become a very popular tool for hyperbolic systems of conservation and balance laws after the pioneer work of Nessyahu and Tadmor, [38], where a second-order, shock-capturing, finite-volume central scheme on a staggered grid was proposed. Since 1990, several higher-order and multidimensional extensions and generalizations of staggered central schemes have been introduced (see, e.g., [40] and references therein). However, staggered central schemes may not provide a satisfactory resolution when small time steps are enforced by stability restrictions, which may occur, for example, in the application of these schemes to convection-diffusion

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problems as observed in Kurganov and Tadmor [30]. These disadvantages are caused by the accumulation of numerical dissipation. Staggered central schemes can be improved by using some characteristic information on local speeds of propagation. This leads to a class of central-upwind schemes developed by Kurganov et al. in [25–27,30]. The central-upwind schemes are based on (one-sided) local speeds which represent the extreme eigenvalues of the system. The use of these techniques makes the central-upwind schemes less dissipative compared to the staggered central schemes, and at the same time being Riemann-problem-solver-free methods they retain the major advantage of central schemes-simplicity. The central-upwind schemes have been successfully applied to a variety of problems including several shallow water models [4,6,12,13,23,24,28,29,32].

SWEs and related models are of great interest for many atmospheric and oceanic applications as well as for modeling flows in the rivers and coastal areas. To be able to accurately model realistic situations, one has to develop numerical methods on unstructured grids due to their flexibility to represent irregular domains and convenience of local mesh refinement. There are two main proprieties a good numerical method for SWEs should satisfy. The first one is called a *well-balanced property:* the scheme should exactly preserve "lake at rest" steady-state solutions. The second property is *positivity preserving:* the method should guarantee positivity of the computed values of the water depth in each point of the domain at all times.

The main widely used unstructured finite-volume methods are *cell-centered* (CCFVM) and *cell-vertex* (CVFVM) ones. The cell-vertex methods are sometimes referred to as node-centered, mesh-vertex or vertex-centered methods. For the CCFVM, the cells are the triangles of the primary mesh. For CVFVM the cells are the dual of the primary mesh as explained in the next section. For a detailed discussion on the two methods we refer the reader to [3,34,36].

Diskin et al. [17] have compared the node-centered and cell-centered schemes for finite-volume discretization of Poisson's equation as a model with viscous fluxes. They have tested structured and unstructured grids based on both triangular and quadrilateral computational cells with randomly perturbed grid points. The authors found that the node-centered finite-volume methods typically outperform the cell-centered ones in terms of accuracy and convergence when the same number of degrees of freedom is used.

Nikolos and Delis [39] proposed a cell-vertex upwind scheme for shallow water flows with wet/dry fronts over complex bottom topography. The authors used the Roe method to compute numerical fluxes and the time evolution of their scheme was carried out by an explicit four-stage Runge-Kutta method. Delis et al. [16] have recently performed an extensive comparison between node-centered and cell-centered upwind finite-volume methods for the 2D SWEs with different source terms on unstructured grids. They studied the performance, robustness and defectiveness of the two methods by comparing numerical results with both analytical solutions and experimental and field data. In the analyses, the authors used different structures of computational grids and the comparisons were performed for all conservative variables using different norms. They found that the CVFVM leads to identical convergence behavior for grids with various qualities (in terms of orientation and distortion) while in the CCFVM, the results are influenced by the grid quality. The reason is that the cells in the CVFVM are constructed in a way that leads to more spatial uniformity than the CCFVM. The authors concluded in their analyses that the CCFVM require more attention in order to obtain an appropriate correction in the construction of the extrapolated primitive variables of the system. Without adequate correction, the points where the numerical fluxes are evaluated do not correspond to the flux vectors obtained by extrapolations. The quality of the results of the CVFVM are less affected by the grid geometry. In addition to the advantages already mentioned, the CVFVM present an advantage compared to the CCFVM for the treatment of the boundary conditions since in the case of the CVFVM the control volume centers can be located on the boundary of the computational domain.

To the best of our knowledge, no cell-vertex central-upwind scheme for shallow water flows or hyperbolic systems of conservation laws have been proposed in the literature. Bryson et al. [6] have proposed a central-upwind scheme on triangular grids for the Saint-Venant system of SWEs with possibly discontinuous bottom topography. The authors have showed that their method is well-balanced and positivity preserving, and demonstrated high resolution and robustness of the method. In this paper, we introduce a new well-balanced positivity preserving central-upwind scheme on *cell-vertex* grids (described in Section 2.1) for the 2D SWEs with variable topography.

The paper is organized as follows. In Section 2, we present the new cell-vertex semi-discrete central-upwind scheme for the SWEs (1). In Section 3, we propose a positivity preserving reconstruction for water surface elevation. The well-balanced discretization of the source term is developed in Section 4. The positivity preserving property of the proposed scheme is proved in Section 5. In Section 6, we demonstrate the high resolution and robustness of the proposed method by testing it on a variety of numerical experiments. The final Section 7 contains concluding remarks.

2. The cell-vertex central-upwind scheme

In this section, we focus on the derivation of the proposed cell-vertex central-upwind scheme. First the cell-vertex unstructured grid and the notations used in this paper are described in Section 2.1. Then, we develop the central-upwind method over cell-vertex grids for the SWEs (1), which can be rewritten using the vector of variables $\boldsymbol{U} := (w, p, q)^T$ as

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U}, B)_x + \boldsymbol{G}(\boldsymbol{U}, B)_y = \boldsymbol{S}(\boldsymbol{U}, B) \tag{2}$$

with

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{U}, B) &= \left(p,\ \frac{p^2}{w-B} + \frac{g}{2}(w-B)^2,\ \frac{pq}{w-B}\right)^T,\\
\boldsymbol{G}(\boldsymbol{U}, B) &= \left(q,\ \frac{pq}{w-B},\ \frac{q^2}{w-B} + \frac{g}{2}(w-B)^2\right)^T,\\
\boldsymbol{S}(\boldsymbol{U}, B) &= \left(0,\ -g(w-B)B_x,\ -g(w-B)B_y\right)^T,
\end{aligned} \tag{3}$$

where $w := h + B$ represents the water surface elevation and $p := hu$ and $q := hv$ denote the discharges in the x- and y-directions, respectively.

2.1. Cell-vertex grid and notations

Unstructured cell-vertex grids are obtained using a triangular discretization of the global domain $\mathcal{D}$: The finite-volume cells, denoted by M_j, are centered around the vertices as shown in Fig. 1. There are various methods to define the dual grid. The control volume around each node can be defined by connecting either the barycenters [39] or centroids [21] of the surrounding triangles of the node. These points can be connected either directly or with the midpoints of the edges that meet the node. In this paper, the boundary ∂M_j of the cell M_j around each internal triangulation vertex P_j is defined by connecting directly the centers of mass of the surrounding triangles that have P_j as a common vertex. The water surface elevation w and the discharges p and q are then represented by the corresponding cell averages over the cells M_j of size $|M_j|$ with the centers of mass denoted by $G_j \equiv (x_j, y_j)$.

We assume that the discretization $\mathcal{D} = \bigcup_{j=1}^{N} M_j$ consists of N non-overlapping cells (N is equal to the number of nodes of