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# High-order shock-capturing hyperbolic residual-distribution schemes on irregular triangular grids

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ABSTRACT

In this paper, we construct second- and third-order non-oscillatory shock-capturing hyperbolic residual-distribution schemes for irregular triangular grids, extending the schemes developed in J. Comput. Phys., 300 (2015), 455–491 to discontinuous problems. We present extended first-order N- and Rusanov-scheme formulations for a hyperbolic advection-diffusion system, and demonstrate that the hyperbolic diffusion term does not have any adverse effect on the solution of inviscid problems for a vanishingly small viscous coefficient. We then construct second- and third-order non-oscillatory hyperbolic residual-distribution schemes by blending the non-monotone second- and third-order schemes with the extended first-order schemes as typically done in the residual-distribution schemes, and examine them for discontinuous problems on irregular triangular grids. We also propose to use the Rusanov scheme to avoid non-physical shocks in combination with an improved characteristics-based nonlinear wave sensor for detecting shocks, compression, and expansion regions. We then verify the design order of accuracy of these blended schemes on irregular triangular grids.



## 1. Introduction

Accurate detection of discontinuities are of great interest to many practical applications. Equally, accurate prediction of solution and solution gradients in the smooth regions on irregular grids are also essential in estimating many important physical quantities such as viscous stresses, vorticity, and heat flux. In Ref. [1], we presented new second- and third-order hyperbolic advection-diffusion Residual-Distribution (RD) schemes called the RD-CC2 and RD-CC3 schemes, respectively, and demonstrated that these schemes predict solution and solution gradients efficiently and accurately on anisotropic and irregular triangular grids. These schemes are constructed based on the hyperbolic method [2], where the diffusion term is formulated as a hyperbolic system by including the solution gradients as extra variables, but with a new design principle that ensures the cell residual vanishes for exact quadratic (for second-order) and cubic (for third-order) solutions for arbitrary triangular elements. The new design principle was proposed in Ref. [1] and found critical for smooth and accurate predictions of solution gradients both on the physical geometry and within the domain for highly irregular elements. These schemes also produce solution and solution gradients with an equal order of accuracy on fully irregular elements. These schemes, however, produce oscillatory solution around discontinuities (such as a shock) and therefore, some special treatment is needed to prevent such oscillatory solutions. The objective of the work presented in this paper is, therefore, to develop non-oscillatory versions of the RD-CC2 and RD-CC3 schemes with a mechanism to avoid entropy-violating shocks.

Before discussing strategies for constructing non-oscillatory schemes, we remark that the RD-CC2 and RD-CC3 schemes have an unconventional feature compared with other RD schemes. These schemes have the advective term coupled with the diffusive term even in the advection limit via the extra variables introduced to formulate diffusion as a hyperbolic system. In other words, vanishingly small diffusion coefficient results in a pure advection scheme coupled with the equations for the extra variables. The coupling is advantageous becasue it helps improve the order of accuracy of the advection scheme, typically, by one order [1]. However, a conventional non-oscillatory technique developed for the advection equation, which does not take into account the coupling, may not be immediately applicable to the RD-CC2 and RD-CC3 schemes. For these reasons, we consider the advection-diffusion equation throughout the paper although our target problems are purely advective, and seek a non-oscillatory technique that is simple and easy to apply to these schemes.

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For the construction of non-oscillatory schemes, few approaches have been proposed that are widely used within the RD community. These are: (1) nonlinear advection schemes such as the modified N-scheme or the Positive-Streamwise-Invariant (PSI) scheme [3], and limited schemes [4], where a high-order smooth solution is recovered from a first-order positive scheme with a smoothness indicator, and (2) blended schemes [5], in which a first-order and high-order schemes are blended through a nonlinear blending function. Although these approaches are different, one may recover an identical scheme from either of these approaches [6,7]. These nonlinear schemes are first developed for the scalar advection equation and later extended for a system of equations [8,9]. These schemes are widely used within the RD community [6,7,10–12], and applied to advection and advection-diffusion [13–16], steady inviscid [12,17,18], steady Navier–Stokes [19], turbulent compressible flows [20], and unsteady [12,21,22] problems. It may also be possible to employ an artificial viscosity technique as widely used in the stabilized finite-element methods, e.g., Refs. [23,24], because the RD schemes can be formulated as Petrov–Galerkin schemes.

Among various approaches, in this paper, we consider the blending approach [5,7,12] for constructing non-oscillatory RD-CC2 and RD-CC3 schemes. Specifically, we construct a monotone first-order scheme for the hyperbolic advection-diffusion system by applying first-order RD schemes known to be monotone for hyperbolic systems, the N-scheme and the Rusanov scheme, and then blend it with the RD-CC2 and RD-CC3 schemes through a nonlinear blending function similar to the one presented in, e.g., Ref. [5]. This strategy has been found simple, systematic, and also convenient as entropy-violating shocks can be avoided within the same framework. Other approaches may also be explored, but the comparison of different approaches is beyond the scope of the paper and thus left as future work.

We also propose a technique to avoid entropy-violating shocks by activating the first-order Rusanov scheme at sonic expansion. This approach requires accurate detection of sonic expansion. We therefore, perform this task by developing a new characteristics-based nonlinear wave sensor to accurately detect sonic expansion. The presented technique is an improvement to the technique reported in Refs. [25,26], which uses divergence of the steady characteristics as a mechanism to identify whether an element is in a shock, rarefaction, or away from such nonlinear waves. The technique of Refs. [25,26], however, requires a threshold, and that is often very difficult to know a priori; a large threshold causes instability by high-order methods, while small thresholds lead compression waves to be treated as shocks, which in turn make the solution less accurate and undesirable. Here, we improve this technique with a more accurate characteristics-based nonlinear wave operator that is less dependent on such thresholds. In the present work, the proposed characteristic-based sensor is used as an alternative approach to a more traditional entropy fix technique [7,27] for avoiding unphysical shocks (entropy-violating solutions). Another alternative is to use a special quadrature formula [26], but this technique requires development of completely new high-order schemes and therefore, is not pursued in this study. The proposed sensor may also be used as a first step toward the development of a shock-fitting scheme [28–30].

In this paper, we focus on two-dimensional hyperbolic advection-diffusion systems and develop second- and third-order blended hyperbolic residual-distribution schemes for discontinuous problem on irregular triangular grids. We first demonstrate that the hyperbolic diffusion terms do not negatively affect the solution of the advection equation as the diffusion coefficient approaches zero. We then demonstrate that these blended schemes can successfully detect physical discontinuities using the developed characteristics-based nonlinear wave sensor, and avoid unphysical shocks when the proposed extended Rusanov scheme is used as a first-order advection-diffusion scheme. Through numerical examples, we show that the proposed schemes not only provide an accurate solution but also give accurate and smooth solution gradients (away from discontinuities) on irregular grids. This is extremely important because, as we will demonstrate, least squares reconstruction of gradients, which is commonly used, could be very inaccurate and oscillatory even if a high-order solution is used as a basis for the gradient reconstruction.

The paper is organized as follows. In Section 2, we briefly describe the basics of a nonlinear hyperbolic advection-diffusion system. In Section 3, we present extended first-order N- and Rusanov-schemes for a general hyperbolic advection-diffusion system. In Section 4, we review the baseline hyperbolic RD, and the second- and third-order hyperbolic RD schemes, proposed in Ref. [1]; we use these schemes for the construction of high-order blended schemes, which are presented in Section 5. In Section 6, we discuss how entropy-violating solutions can be avoided, followed by a boundary condition formulation in Section 7. We then present numerical examples in Section 8, demonstrating the shock-capturing capability of the constructed second- and third-order blended hyperbolic advection-diffusion RD schemes on irregular triangular grids. Order of accuracy of these blended schemes is also verified in this section. We then summarize the presented work with some concluding remarks in Section 9.

## 2. General nonlinear hyperbolic advection-diffusion system and discretization

Consider the following general two-dimensional nonlinear advection-diffusion equation:

$$\partial_t u + \partial_x f + \partial_y g = \partial_x(\nu \partial_x u) + \partial_y(\nu \partial_y u) + \tilde{s}(x, y, u), \tag{1}$$

where $f$ and $g$ are nonlinear functions of $u$, $\nu = \nu(u)$, and $\tilde{s}(x, y, u)$ denotes a source term. In this work, we only consider a constant diffusion coefficient, but the schemes are presented in the form directly applicable to nonlinear diffusion coefficient. The advection speeds in $x$ and $y$ directions are therefore $a(u) = \partial f/\partial u$ and $b(u) = \partial g/\partial u$, respectively. We reformulate the advection-diffusion equation in the form of a nonlinear hyperbolic advection-diffusion system using a preconditioning matrix $\mathbf{P}$, which is to simplify the construction of the numerical scheme [31]:

$$\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial \tau} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{Q}, \tag{2}$$

where $\mathbf{U} = [u, p, q]^T$, where the superscript $T$ indicates the transpose, and

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & T_r/\nu(u) & 0 \\ 0 & 0 & T_r/\nu(u) \end{bmatrix},$$

$$\mathbf{F} = \mathbf{F}^a + \mathbf{F}^d = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -p \\ -u \\ 0 \end{bmatrix}, \tag{3}$$

$$\mathbf{Q} = \mathbf{Q}^s + \mathbf{Q}^d = \begin{bmatrix} s(x, y, u) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -p/\nu(u) \\ -q/\nu(u) \end{bmatrix},$$

$$\mathbf{G} = \mathbf{G}^a + \mathbf{G}^d = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -q \\ 0 \\ -u \end{bmatrix}, \tag{4}$$

where $\tau$ is the pseudo-time, and $T_r = L^2/\nu$ is the relaxation time with length scale defined as $L = 1/2\pi$. In the pseudo-steady state,