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A high-order radial basis function (RBF) Leray projection method for the solution of the incompressible unsteady Stokes equations

Edward J. Fuselier[a], Varun Shankar[b,*], Grady B. Wright[c]

[a] Department of Mathematics and Computer Science, High Point University, NC, USA
[b] Department of Mathematics, University of Utah, UT, USA
[c] Department of Mathematics, Boise State University, ID, USA

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ABSTRACT

A new projection method based on radial basis functions (RBFs) is presented for discretizing the incompressible unsteady Stokes equations in irregular geometries. The novelty of the method comes from the application of a new technique for computing the Leray–Helmholtz projection of a vector field using generalized interpolation with divergence-free and curl-free RBFs. Unlike traditional projection methods, this new method enables matching both tangential and normal components of divergence-free vector fields on the domain boundary. This allows incompressibility of the velocity field to be enforced without any time-splitting or pressure boundary conditions. Spatial derivatives are approximated using collocation with global RBFs so that the method only requires samples of the field at (possibly scattered) nodes over the domain. Numerical results are presented demonstrating high-order convergence in both space (between 5th and 6th orders) and time (up to 4th order) for some model problems in two dimensional irregular geometries.

1. Introduction

Radial basis functions (RBFs) are increasingly used as the building blocks of methods for the numerical solution of partial differential equations (PDEs), primarily due to their ability to discretize differential operators on arbitrary geometries using scattered node layouts. The potential for spectral accuracy of global RBFs on smooth problems further adds to their appeal. These methods have been used to solve PDEs on planar regions [5,12,22,23,32], spherical regions [7,8,19,37], and general surfaces [17,31].

In this work, we present a novel high-order RBF collocation method for the numerical solution of the unsteady (or time-dependent) incompressible Stokes equations on some (irregular) 2D domain Ω. These equations are obtained in the zero Reynolds number limit of the incompressible Navier–Stokes equations and are given by

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \Delta \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where $\boldsymbol{u} = [u \quad v]^T$ is the velocity field, p is the pressure, $\boldsymbol{f}$ is some forcing term, ρ is the (constant) fluid density, and ν is the coefficient of kinematic viscosity. We consider no-slip conditions on the boundary of Ω, denoted by $\partial\Omega$,

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{g}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega, \tag{2}$$

with the restriction that $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n} = 0$, where $\boldsymbol{n}$ is the respective outward normal unit vector on $\partial\Omega$.

One of the dominant approaches to numerically solving the unsteady Stokes equations (and by extension, the Navier–Stokes equations) is to use so-called fractional-step or projection methods, which were first introduced independently by Chorin [4] and Temam [34] and have advanced to more modern exact and approximate projection methods based on pressure Poisson equations; for example, see [3,25,26,39]. Broadly speaking, these methods employ operator splitting, and use the pressure to *project* an intermediate velocity field to the space of incompressible or divergence-free velocity fields. Such methods are more efficient than methods that solve the saddle-point-like systems that arise from the discretization of (1) (for more, see [2]). However, they typically require specialized grids, and the careful selection of pressure and intermediate velocity boundary conditions to match the actual boundary conditions on the velocity field; indeed, these methods typically only match the normal components, and attempt to match tangential components through time extrapolation [3]. As a result of operator splitting, it is difficult (and in many cases impossible) to attain high orders of accuracy for the velocity field in

* Corresponding author. Tel.: +1 3852598400.
E-mail addresses: efuselie@highpoint.edu (E.J. Fuselier), vshankar@math.utah.edu (V. Shankar), gradywright@boisestate.edu (G.B. Wright).

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time. In addition, errors in computing the pressure may propagate to the velocity field, since the pressure is often used to project the intermediate velocity field. For an overview of projection methods for incompressible flows, see [20].

The RBF method we develop for the unsteady Stokes equations also employs a projection-based approach, but avoids the issues detailed above. This is achieved by developing a numerical approximation to the projection operator that can be applied to (1) in such a way as to avoid any errors from operator splitting. The method relies on the following alternative form of (1) originally proposed by Leray [9,24]:

$$\frac{\partial \boldsymbol{u}}{\partial t} = \Pi\left(\nu \Delta \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{f}\right), \tag{3}$$

where the operator Π is the so-called Leray–Helmholtz (or simply Leray) projector and has the property that for a vector field $\mathbf{w}$ on a connected domain Ω, $\Pi(\mathbf{w}) = \mathbf{v}$ where $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \boldsymbol{n} = 0$. The existence of this operator follows from the Helmholtz–Hodge decomposition theorem. Eq. (3) is simply a forced diffusion-type equation for $\boldsymbol{u}$ and is obtained by applying Π to the first equation in (1) and using the fact that $\Pi(\nabla p) = 0$. Using the orthogonal complement of Π, which we denote as $\Pi^{\perp}$, the following auxiliary equation to (3) for the gradient of the pressure can be obtained:

$$\frac{1}{\rho}\nabla p = \Pi^{\perp}\left(\nu \Delta \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{f}\right). \tag{4}$$

This allows the pressure to be recovered as an instantaneous functional of the velocity field. Our goal is to construct a discrete approximation to Π and solve (3) using collocation and the method-of-lines. We will use this to recover the pressure as in (4).

To discretize the Leray projector, we adopt the recent method from [18] for computing the Helmholtz–Hodge decomposition of a vector field based on generalized RBF interpolation with matrix-valued kernels. This method provides a well-posed way to construct a discrete Leray projector over a set of scattered nodes X in some domain Ω, with the exact normal boundary conditions enforced at points along the domain boundary. It can then be applied to any vector field sampled at X to recover an *analytically* divergence-free field everywhere on interior and boundary of Ω. We modify the technique of [18] here so that Π also incorporates tangential boundary conditions, unlike both the traditional Leray projector and the implicit Leray projection performed by methods that use the pressure Poisson equation. An approximation to $\Pi^{\perp}$ can also be obtained trivially from this technique, and we use that to recover the pressure p, but without solving a differential equation for p . We combine the discrete Leray projection with a global RBF collocation method for approximating the Laplacian in (3) to get a semi-discrete approximation, which we then integrate in time with a high-order BDF scheme. The method avoids any errors from operator splitting, does not require the use of specialized grids or meshes, and can be used with scattered nodes. We demonstrate that this new methodology allows for high order accuracy in space and arbitrary orders of convergence in time on irregular domains.

Similar approaches to the one we propose have been adopted with divergence-free RBFs [36], divergence-free wavelets [21], and divergence-free polynomials in the context of finite elements [38]. The first approach uses a similar matrix-valued kernel, but enlarges it to account for the pressure coupling, which increases the computational cost. It also is presently restricted to time-independent problems. The wavelet approach is similar to ours in that it employs the (standard) Leray projection, but it is restricted to far more regular domains and nodes (rectangular boxes with tensor product nodes). The finite element approach differs from ours in that a weak formulation with divergence-free test functions is employed to remove the pressure. This method also requires a mesh, and boundary conditions are only enforced weakly.

The remainder of the paper is organized as follows. In Section 2, RBF collocation for approximating the Laplacian is briefly reviewed. In Section 3, generalized interpolation of vector-valued functions with matrix-valued RBFs is presented and applied to the construction of a discrete Leray projector. Equipped with spatial discretizations of the Laplacian and the Leray projector, we then discuss the full space-time discretization of the incompressible unsteady Stokes equations in Section 4. Eigenvalue stability of the method is investigated in Section 5. Numerical results demonstrating the spatial and temporal accuracy of our method are given in Section 6 using two problems involving time-dependent boundary conditions: rotational flow on a rotating disk, and a flow on a rectangular domain containing a spinning disk in its interior. This is followed by a summary of the method and a discussion of future enhancements in Section 7.

2. RBF collocation method for the Laplacian

Let $\Omega \subseteq \mathbb{R}^d$, and $\phi : \Omega \times \Omega \to \mathbb{R}$ be a kernel with the property $\phi(\boldsymbol{x}, \boldsymbol{y}) := \phi(\|\boldsymbol{x} - \boldsymbol{y}\|)$ for $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, where $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^d$. We refer to kernels with this property as *radial kernels* or *radial functions*. Given a set of nodes $X = \{\boldsymbol{x}_k\}_{k=1}^N \subset \Omega$ and a continuous target function $f : \Omega \to \mathbb{R}$ sampled at the nodes in X, the standard RBF interpolant to the data has the form

$$s_f(\boldsymbol{x}) = \sum_{k=1}^{N} c_k \phi(\|\boldsymbol{x} - \boldsymbol{x}_k\|). \tag{5}$$

The expansion coefficients $\{c_k\}_{k=1}^N$ are determined by enforcing $s|_X = f|_X$. This can be expressed as the following linear system:

$$\underbrace{\begin{bmatrix} \phi(r_{1,1}) & \phi(r_{1,2}) & \dots & \phi(r_{1,N}) \\ \phi(r_{2,1}) & \phi(r_{2,2}) & \dots & \phi(r_{2,N}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(r_{N,1}) & \phi(r_{N,2}) & \dots & \phi(r_{N,N}) \end{bmatrix}}_{A_X} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}}_{c_f} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}}_{f_X}, \tag{6}$$

where $r_{i,j} = ||\boldsymbol{x}_i - \boldsymbol{x}_j||$. If ϕ is, for example, a positive-definite radial kernel on $\mathbb{R}^d$, and all nodes in X are distinct, then the matrix A_X above is guaranteed to be positive definite, hence (5) is well-posed. Examples of various choices for ϕ, including relaxed conditions to guarantee the well-posedness of (5) can be found in [5, Chap. 3–12]). The convergence of these RBF interpolants to the target function as the number of samples increases is, in general, determined by the rate of decay of the Fourier transform of ϕ. If the rate of decay is algebraic, then one can expect algebraic convergence for a sufficiently smooth function. If instead the Fourier transform decays exponentially, then one can expect exponential (or spectral) convergence for a large class of infinitely smooth functions.

The RBF interpolant (5) can be exploited to construct discrete approximations to linear differential operators (differentiation matrices) in the same fashion as standard Chebyshev or Fourier collocation methods [12, Chap. 3–4]. However, the RBF method is not restricted to special node sets. Let $\mathcal{L}$ be a linear differential operator of interest and suppose we wish to approximate $\mathcal{L}f$ using the RBF interpolant (5) of f sampled at X. Applying $\mathcal{L}$ to (5) and evaluating at the points in X gives the following matrix equation:

$$\begin{bmatrix} (\mathcal{L}s_f)_1 \\ (\mathcal{L}s_f)_2 \\ \vdots \\ (\mathcal{L}s_f)_N \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{L}\phi(r_{1,1}) & \mathcal{L}\phi(r_{1,2}) & \dots & \mathcal{L}\phi(r_{1,N}) \\ \mathcal{L}\phi(r_{2,1}) & \mathcal{L}\phi(r_{2,2}) & \dots & \mathcal{L}\phi(r_{2,N}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}\phi(r_{N,1}) & \mathcal{L}\phi(r_{N,2}) & \dots & \mathcal{L}\phi(r_{N,N}) \end{bmatrix}}_{A_X^{\mathcal{L}}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}}_{c_f}, \tag{7}$$