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A well-balanced positivity-preserving central-upwind scheme for shallow water equations on unstructured quadrilateral grids

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ABSTRACT

We introduce a new second-order central-upwind scheme for shallow water equations on the unstructured quadrilateral grids. We propose a new technique for bottom topography approximation over quadrilateral cells as well as an efficient water surface correction procedure which guarantee the positivity of the computed fluid depth. We also design a new quadrature for the discretization of the source term, using which the new scheme exactly preserves "lake at rest" steady states. We demonstrate these features of the new scheme as well as its high resolution and robustness and its potential advantages over the triangular central-upwind scheme in a number of numerical examples.

1. Introduction

In this paper we consider the two-dimensional (2D) shallow water equations (SWEs):

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x, \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghB_y. \end{cases} \tag{1}$$

Here, $h(x, y, t)$ is the water depth, $u(x, y, t)$ and $v(x, y, t)$ are the $x-$ and $y-$ velocities, respectively, $B(x, y)$ is bottom topography and g is the gravitational constant. 2D SWEs are commonly used to simulate a wide range of problems in water resources engineering, modeling oceans, rivers and coastal areas, etc.

The system (1) admits several steady-state solutions. One of the practically most important steady states is a so-called "lake at rest" state satisfying,

$$u \equiv v \equiv 0, \;\; h + B = const. \tag{2}$$

A good numerical method for the SWEs (1) should be well-balanced, that is, it should be capable to exactly preserve the "lake at rest" steady states (2). It should also preserve positivity of the water depth h.

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Many numerical methods for SWEs were developed in past decades. We refer the reader, for example, to finite difference [1–4], finite element [4–9] and finite volume [4,10–15] methods. In this paper we focus on the finite volume method which are based on the integral form of (1) and thus are naturally designed to conserve the mass.

Central upwind scheme is one of the finite volume methods that is both well-balanced and positivity preserving. Central-upwind schemes are Riemann-problem-solver-free Godunov-type methods that were originally introduced in [16] for the general multidimensional systems of hyperbolic conservation law and further developed in [17–20]. In [21,22], the central- upwind scheme for the SWEs were developed in the one-dimensional (1D) and 2D cases using Cartesian grids. In [23], the central-upwind schemes were extended to unstructured triangular meshes, and in [24], they were also generalized for polygon cell-vertex meshes.

The main goal of this paper is to develop a second-order well-balanced positivity preserving central-upwind scheme for (1) on unstructured quadrilateral grids. Such grids have been widely used in finite volume methods for various applications, in particular, for numerically solving incompressible Navier–Stokes, diffusion equations, semilinear elliptic and elliptic systems, see, e.g., [25–28] and references therein. In particular, quadrilateral grids have been used to develop finite volume methods for the 2D SWEs, see, e.g., [29–34]. Unstructured quadrilateral grids are popular since they allow one to relatively easily implement the local and adaptive mesh refinement techniques [35,36], increase the formal order of spatial accuracy of the scheme, and discretize the second- and higher-order terms [30,37]. Comparing to the triangular grids, one of the main advantages of the quadrilateral ones is that quadrilateral cells have more neighboring cells and thus the quadrilateral time evolution procedure is typically more accurate.

The proposed quadrilateral central-upwind scheme is an extension of the triangular central -upwind scheme from [23]. However, some of the ingredients of the triangular scheme cannot be directly carried to the quadrilateral case. For example, one cannot obtain a continuous piecewise linear approximation of the bottom topography. Instead, we introduce a new bottom topography approximation. In each quadrilateral cell the bottom topography function B is replaced with four continuous linear pieces, each of which connects the values of B at two of the neighboring cell vertices with the approximation value of B at the geometric center of the cell. Another novelty of our quadrilateral scheme is a new water surface reconstruction correction technique, required to guarantee the positivity of the water depth at the reconstruction step of the central-upwind scheme. To this end, we first perform a piecewise linear reconstruction of the water surface and then, in the cells where some values of the reconstructions fall below the corresponding values of the bottom topography, we replace the linear piece with four continuously matched linear pieces whose shape is similar to the bottom topography approximant in this cell. As we prove in Theorem 1, this guarantees the positivity of the water depth h. To ensure the well-balanced property of the proposed scheme, we design a special quadrature for the cell average of the geometric source term, which leads to a perfect balance of the source and fluxes for the "lake at rest" state.

To the best of our knowledge, the designed central-upwind scheme is among the first well-balanced positivity preserving schemes on unstructured quadrilateral grids.

The paper is organized as follows. The proposed central-upwind scheme is described in Section 2 and its well-balanced and positivity preserving properties are proved in Sections 3 and 4. In Section 5, the new scheme is tested on a number of numerical experiments which demonstrate high accuracy robustness of the proposed scheme and also emphasize its potential advantages over its triangular counterpart. Finally, we finish the paper with concluding remarks in Section 6.

2. Central-upwind cheme on unstructured quadrilateral grids

First, we introduce the water surface variable $w = h + B$ and rewrite the system (1) in the following equivalent form:

$$U_t + F(U,B)_x + G(U,B)_y = S(U,B), \tag{3}$$

where

$$U = (w, hu, hv)^T, \tag{4}$$

$$F(U,B) = \left(hu, \frac{(hu)^2}{w-B} + \frac{g}{2}(w-B)^2, \frac{(hu)(hv)}{w-B} \right)^T, \tag{5}$$

$$G(U,B) = \left(hv, \frac{(hu)(hv)}{w-B}, \frac{(hu)^2}{w-B} + \frac{g}{2}(w-B)^2 \right)^T, \tag{6}$$

$$S(U,B) = (0, -g(w-B)B_x, -g(w-B)B_y)^T. \tag{7}$$

Let the computational domain discretization $\Omega = \cup_{j=1}^{N} E_j$ be covered by a quadrilateral grids with the cells E_j of size $|E_j|$. A typical cell E_j together with its neighbours $E_{jk},\ k = 1,2,3,4$ are outlined in Fig. 1.

We denote by $\vec{n}_{jk} := (\cos(\theta_{jk}), \sin(\theta_{jk}))$ the outer unit normals of the corresponding sides of E_j of length $l_{jk},\ k = 1,2,3,4$. The coordinates of the geometric center (center of mass) of the E_j are denoted by (x_j, y_j) and $M_{jk} := (x_{jk}, y_{jk}),\ k = 1,2,3,4$ is the midpoint of the k−th side of the quadrilateral E_j.

In the semi-discrete central-upwind scheme, the cell average of the computed solutions, $\bar{U}_j^{(t)} \approx \frac{1}{E_j}\int_{E_j}^{U}(x,y,t)dxdy$, are evolved in time

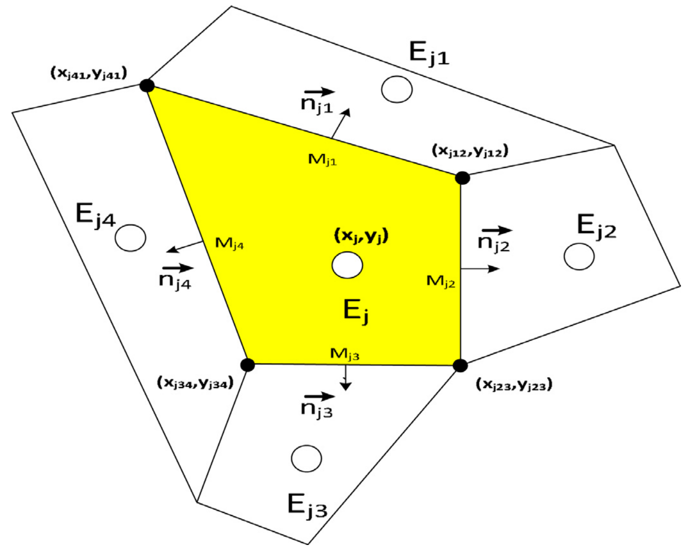

Fig. 1. An unstructured quadrilateral cell with its four neighboring cells.

by solving the following system of ODEs:

$$\begin{aligned}\frac{d\bar{U}_j}{dt} = &-\frac{1}{|E_j|}\sum_{k=1}^{4}\frac{l_{jk}\cos(\theta_{jk})}{a_{jk}^{in}+a_{jk}^{out}}\Big[a_{jk}^{in}F\big(U_{jk}(M_{jk}),B(M_{jk})\big)\\ &+a_{jk}^{out}F\big(U_j(M_{jk}),B(M_{jk})\big)\Big]\\ &-\frac{1}{|E_j|}\sum_{k=1}^{4}\frac{l_{jk}\sin(\theta_{jk})}{a_{jk}^{in}+a_{jk}^{out}}\Big[a_{jk}^{in}G\big(U_{jk}(M_{jk}),B(M_{jk})\big)\\ &+a_{jk}^{out}G\big(U_j(M_{jk}),B(M_{jk})\big)\Big]\\ &+\frac{1}{|E_j|}\sum_{k=1}^{4}l_{jk}\frac{a_{jk}^{in}a_{jk}^{out}}{a_{jk}^{in}+a_{jk}^{out}}\big[U_{jk}(M_{jk})-U_j(M_{jk})\big]+\bar{S}_j,\end{aligned} \tag{8}$$

which can be derived similarly to the derivation procedure proposed for a triangular grids in [20,23]. Notice that all the indexed quantities in (8) are functions of t, but from now on we omit this dependence for the sake of brevity.

The values $U_j(M_{jk})$ and $U_{jk}(M_{jk})$ are the values at M_{jk} of the two polynomial pieces reconstructed in cells $E_j$and E_{jk}, respectively. The corresponding piecewise linear reconstruction is:

$$\tilde{U}_j(x,y) = \bar{U}_j + (U_x)_j(x-x_j) + (U_y)_j(y-y_j). \tag{9}$$

To minimize the oscillations, the slopes $(U_x)_j$ and $(U_y)_j$ are to be computed using a nonlinear limiter. We propose the following minmod-type limiter which will be applied in a component wise manner. Consider the ith component of U, we first construct four linear interpolations $L_j^{12}, L_j^{23}, L_j^{34}$ and L_j^{41}, each of which is obtained by considering the three points at the geometric center of E_j and corresponding two neighboring cells. For example, L_j^{12} is obtained by passing the plane through $(x_j, y_j, \bar{U}_j^{(i)})$, $(x_{j1}, y_{j1}, \bar{U}_{j1}^{(i)})$ and $(x_{j2}, y_{j2}, \bar{U}_{j2}^{(i)})$. Notice that all of the four obtained interpolants are conservative in the cell E_j by construction. We then select the linear piece with the smallest magnitude of the gradient, say L_j^{km}, and we set:

$$\left((U_x)_j^{(i)}, (U_y)_j^{(i)}\right) = \nabla L_j^{km}. \tag{10}$$

In order to further minimize the reconstruction oscillations, the reconstructed values calculated at the points $M_{jk},\ k = 1,2,3,4$ are checked. If the reconstructed value of $U_j^{(i)}(M_{jk})$ is not between the cell averages $\bar{U}_j^{(i)}$ and $\bar{U}_{jk}^{(i)}$, we set

$$\left((U_x)_j^{(i)}, (U_y)_j^{(i)}\right) = 0. \tag{11}$$