Computers & Fluids 107 (2015) 189–195

Contents lists available at ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

One-dimensional linear advection–diffusion equation: Analytical and finite element solutions

Abdelkader Mojtabi [a,*], Michel O. Deville [b]

[a] Université Paul Sabatier and IMFT, 1 Avenue du Professeur Camille Soula, 31400 Toulouse, France
[b] École Polytechnique Fédérale de Lausanne, School of Engineering, Room MXD 130, Station 12, 1015 Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 19 March 2014
Received in revised form 14 October 2014
Accepted 4 November 2014
Available online 18 November 2014

Keywords:
Separation of variables
Finite element method
Exponential layer
Symbolic computation
Asymptotic development

ABSTRACT

In this paper, a time dependent one-dimensional linear advection–diffusion equation with Dirichlet homogeneous boundary conditions and an initial sine function is solved analytically by separation of variables and numerically by the finite element method. It is observed that when the advection becomes dominant, the analytical solution becomes ill-behaved and harder to evaluate. Therefore another approach is designed where the solution is decomposed in a simple wave solution and a viscous perturbation. It is shown that an exponential layer builds up close to the downstream boundary. Discussion and comparison of both solutions are carried out extensively offering the numericist a new test model for the numerical integration of the Navier–Stokes equation.

1. Introduction

The numerical integration of the Navier–Stokes equations by standard methods like FXM (Finite X Methods), X being D(ifference), E(lement), V(olume) or by spectral and spectral elements requires a careful design. This is especially true and essential when long time integration ranges are involved as is the case for direct numerical simulation (DNS) or large-eddy simulations (LES) of turbulent flows where the evaluation of time averaged quantities imply very long time series to obtain meaningful information and statistics. Therefore temporal stability and space accuracy are the basic requirements needed to render the algorithms efficient on large scale parallel machines and to extract relevant physical phenomena.

The practitioners of computational fluid dynamics have decomposed the analysis of the complexity and stiffness of the Navier–Stokes equations into simpler problems like the Stokes (linear) equations that embody the difficulties of the space discretization of the velocity and pressure fields and the advection–diffusion problem that is related to the transport character of the non-linear terms. This last class of problems includes the non-linear Burgers equations and the linear advection–diffusion (LAD) equation. In this paper, we will address the one-dimensional LAD equation with homogeneous Dirichlet boundary conditions as this is a meaningful test for established or novel discrete schemes. For high Reynolds number flows the advection is dominating diffusion but the presence of the boundaries imposing no-slip wall conditions complicates the solution of the problem. Boundary layers develop and in most cases influence deeply the flow dynamics. No-slip wall boundary conditions impede the general use of periodic Fourier representation and spectral calculation.

Even though the LAD equation is linear it is difficult to find closed form analytical solution in the literature. Most of the efforts have been devoted to the solution of LAD with an upstream boundary condition and a Robin or Neumann downstream condition. The presence of the gradient condition at the exit of the domain eases the development of the analytical solution. The paper by Pérez Guérrero et al. [10] uses a change of variable to obtain a heat equation which is then solved by a generalized integral transform technique proposed by Cotta [4]. In [13], van Genuchten et al. are able to use a variable transformation that reduces the partial differential equation to an ordinary differential equation the solution of which is expressed by the complementary error function. Other methodologies are possible to tackle the LAD problem on finite or infinite domains. Without the pretension of being exhaustive, we can cite Bosen [3], Kumar et al. [8], Pérez Guérrero et al. [11] and Zoppou and Knight [14].

On the numerical side, finite differences have been applied, see for example [5]. In the finite element framework, Gresho et al. [7] investigate a time integrator based on the combination of the

* Corresponding author.
E-mail addresses: mojtabi@imft.fr (A. Mojtabi), michel.deville@epfl.ch (M.O. Deville).

http://dx.doi.org/10.1016/j.compfluid.2014.11.006
0045-7930/

trapezoid rule and the second-order Adams–Bashforth scheme with piecewise linear elements for space discretization. Some analytical solutions are presented in the various examples solved throughout the paper. However none of them treats the LAD problem with homogeneous Dirichlet conditions and a smooth initial condition like a sine function. In the book of Donea and Huerta [6] the LAD problem is proposed with a truncated Gaussian profile as the initial condition.

In this paper we will solve the LAD problem with homogeneous boundary conditions and a sine profile for the initial condition. This is exactly the same initial and boundary conditions that were imposed for the Burgers equation solved by Basdevant et al. [2]. We will be able to compare the physics associated with both problems. The paper is organized as follows. Section 2 describes the LAD problem which is solved in closed form by the introduction of a change of variables. Section 3 details the analytical solution when the viscosity goes to zero. In this case the problem at hand is a simple wave equation perturbed by the presence of a very weak viscous term. Section 4 presents the Fourier solution when periodic conditions are applied. Section 5 is devoted to some considerations related to energy conservation. Section 6 treats the numerical method obtained by linear finite elements and a time integration using a Crank–Nicolson scheme for the viscous term and a second order Adams–Bashforth scheme for the advection term. Section 7 reports the results produced by both approaches and compares them. Finally the last section draws conclusions.

2. Linear advection–diffusion equation

The unsteady linear advection–diffusion equation is given by the following relation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad -1 < x < 1, \quad t \in]0, T], \tag{1}$$

where u is the velocity variable, $c > 0$ the constant advection velocity, ν the kinematic viscosity and time t. We will impose homogeneous Dirichlet boundary conditions $u(-1,t) = u(1,t) = 0$ and the initial condition $u(x,0) = -\sin\pi x$. These initial and boundary conditions were already used for the Burgers equation in [2]. This choice will allow us to compare the two cases.

To obtain a closed form solution, let us make the change of variables

$$u(x,t) = v(x,t)e^{\alpha x + \beta t}. \tag{2}$$

Introducing (2) in (1) and simplifying by the exponential, one obtains

$$\frac{\partial v}{\partial t} + \left(\beta + c\alpha - \alpha^2\nu\right)v + (c - 2\alpha\nu)\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2}. \tag{3}$$

As α and β are free parameters, we choose them in such a way that

$$\beta + c\alpha - \alpha^2\nu = 0, \tag{4}$$

$$c - 2\alpha\nu = 0. \tag{5}$$

Therefore, $\alpha = c/2\nu$ and $\beta = -c^2/4\nu$. The governing equation for v is reduced to the standard heat equation

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial x^2}, \tag{6}$$

subject to the homogeneous conditions $v(-1,t) = v(1,t) = 0$ and the initial condition

$$v(x,0) = -\sin\pi x e^{-\alpha x} = -\sin\pi x e^{-\frac{cx}{2\nu}}. \tag{7}$$

Let us use the method of separation of variables to solve (6) by setting $v(x,t) = X(x)T(t)$. Omitting the details of the algebra, this leads to the solution

$$v(x,t) = \sum_{k=0}^{\infty}\left(A_k\sin\frac{k\pi x}{2} + B_k\cos\frac{k\pi x}{2}\right)e^{-\nu\frac{k^2\pi^2}{4}t}. \tag{8}$$

The boundary conditions impose the conditions $A_{2p+1} = B_{2p} = 0,\ p = 0, 1, \ldots$. Eq. (8) becomes

$$v(x,t) = \sum_{p=0}^{\infty}A_{2p}\sin(p\pi x)e^{-\nu p^2\pi^2 t} + B_{2p+1}\cos\left(\frac{2p+1}{2}\pi x\right)e^{-\nu\frac{(2p+1)^2}{4}\pi^2 t}. \tag{9}$$

Applying the initial condition (7) to Eq. (9) yields

$$\sum_{p=0}^{\infty}A_{2p}\sin p\pi x + B_{2p+1}\cos\frac{2p+1}{2}\pi x = -\sin(\pi x)e^{-\frac{cx}{2\nu}}. \tag{10}$$

Using the orthogonality property of Fourier polynomials, the coefficients A_{2p} and B_{2p+1} are obtained solving the relations

$$A_{2p}\int_{-1}^{1}(\sin p\pi x)^2 dx = -\int_{-1}^{1}\sin(\pi x)\sin(p\pi x)e^{-\frac{cx}{2\nu}}dx, \tag{11}$$

$$B_{2p+1}\int_{-1}^{1}\left(\cos\frac{2p+1}{2}\pi x\right)^2 dx = -\int_{-1}^{1}\sin(\pi x)\cos\left(\frac{2p+1}{2}\pi x\right)e^{-\frac{cx}{2\nu}}dx. \tag{12}$$

With the help of standard trigonometric relations, the right hand side integral of (11) may be rewritten as

$$\int_{-1}^{1}\sin\pi x\sin p\pi x e^{-\frac{cx}{2\nu}}\,dx = \frac{1}{2}\int_{-1}^{1}[\cos(p-1)\pi x - \cos(p+1)\pi x]e^{-\frac{cx}{2\nu}}dx. \tag{13}$$

Furthermore one has also the identity (cf. [1])

$$\int e^{-ax}\cos p\pi x\,dx = \frac{e^{-ax}}{a^2 + p^2\pi^2}(-a\cos p\pi x + p\pi\sin p\pi x). \tag{14}$$

Therefore one gets

$$A_{2p} = \frac{-32(-1)^{p+1}\nu^3 c\pi^2 p\sinh(c/2\nu)}{c^4 + 8(c\pi\nu)^2(p^2+1) + 16(\pi\nu)^4(p^2-1)^2}. \tag{15}$$

A similar development gives

$$B_{2p+1} = \frac{-16(-1)^{p+1}\nu^3 c\pi^2(2p+1)\cosh(c/2\nu)}{c^4 + (c\pi\nu)^2(8p^2+8p+10) + (\pi\nu)^4(4p^2+4p-3)^2}. \tag{16}$$

With (2) and the relations (9), (15), (16) one writes

$$\begin{aligned} u(x,t) = {} & 16\pi^2\nu^3 c e^{\frac{c}{2\nu}(x-\frac{c}{2}t)} \\ & \times\left[\sinh\left(\frac{c}{2\nu}\right)\sum_{p=0}^{\infty}\frac{(-1)^p 2p\sin(p\pi x)e^{-\nu p^2\pi^2 t}}{c^4 + 8(c\pi\nu)^2(p^2+1) + 16(\pi\nu)^4(p^2-1)^2}\right. \\ & \left. + \cosh\left(\frac{c}{2\nu}\right)\sum_{p=0}^{\infty}\frac{(-1)^p(2p+1)\cos\left(\frac{2p+1}{2}\pi x\right)e^{-\nu\frac{(2p+1)^2}{4}\pi^2 t}}{c^4 + (c\pi\nu)^2(8p^2+8p+10) + (\pi\nu)^4(4p^2+4p-3)^2}\right]. \end{aligned} \tag{17}$$

When the viscosity goes to zero, the solution becomes

$$u(x,t) = 8\pi^2\left(\frac{\nu}{c}\right)^3 e^{\frac{c}{2\nu}((x+1)-\frac{c}{2}t)} \left[\sum_{p=0}^{\infty}(-1)^p\left(2p\sin(p\pi x) + (2p+1)\cos\left(\frac{2p+1}{2}\pi x\right)\right)\right]. \tag{18}$$

We observe that the presence of the exponential term in (18) renders the problem stiffer and the closed form solution blows up for vanishing viscosity. This ill-behavior requires a special treatment.

3. Analytical solution for vanishing viscosity

We will decompose the problem solution in two parts

$$u(x,t) = u_a(x,t) + \nu U(x,t), \tag{19}$$