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Very efficient high-order hyperbolic schemes for time-dependent advection–diffusion problems: Third-, fourth-, and sixth-order

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ABSTRACT

In this paper, we construct very efficient high-order schemes for general time-dependent advection–diffusion problems, based on the first-order hyperbolic system method. Extending the previous work on the second-order time-dependent hyperbolic advection–diffusion scheme (Mazaheri and Nishikawa, NASA/TM-2014-218175, 2014), we construct third-, fourth-, and sixth-order accurate schemes by modifying the source term discretization. In this paper, two techniques for the source term discretization are proposed; (1) reformulation of the source terms with their divergence forms and (2) correction to the trapezoidal rule for the source term discretization. We construct spatially third- and fourth-order schemes from the former technique. These schemes require computations of the gradients and second-derivatives of the source terms. From the latter technique, we construct spatially third-, fourth-, and sixth-order schemes by using the gradients and second-derivatives for the source terms, except the fourth-order scheme, which does not require the second derivatives of the source term and thus is even less computationally expensive than the third-order schemes. We then construct high-order time-accurate schemes by incorporating a high-order backward difference formula as a source term. These schemes are very efficient in that high-order accuracy is achieved for both the solution and the gradient only by the improved source term discretization. A very rapid Newton-type convergence is achieved by a compact second-order Jacobian formulation. The numerical results are presented for both steady and time-dependent linear and nonlinear advection–diffusion problems, demonstrating these powerful features.

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1. Introduction

In this paper, we construct very efficient high-order schemes for general time-dependent advection–diffusion problems, based on the first-order hyperbolic system method [1,2]. In this method, the diffusion term is reformulated as a hyperbolic system, leading to the unification of advection and diffusion as a single hyperbolic system [2]. The drastic change in the type of equations, parabolic to hyperbolic, brings several dramatic improvements in the construction of numerical schemes: hyperbolic schemes for diffusion, the same order of accuracy for the solution and the gradients, orders-of-magnitude convergence acceleration, etc., which have been demonstrated for steady diffusion and viscous problems in Refs. [1–5] and unsteady advection–diffusion problems in Ref. [6]. It is based on the reformulation of the governing equations, and therefore applicable to any discretization method. In this work, we consider a Residual-Distribution (RD) method [7], which has been well developed for hyperbolic systems and has a superior feature of achieving second-order accuracy in a compact stencil.

In the previous work [6], we extended the hyperbolic method, for the first time, to time-accurate computations by an implicit time-integration method based on the second-order backward difference formula. The resulting scheme was applied to various time-dependent problems, demonstrating second-order accuracy for the solution and the gradient achieved at all interior and boundary nodes in uniform and nonuniform grids at every physical time step, and rapid convergence for solving implicit-residual equations by Newton's method (i.e., less than 5 iterations per physical time step), which is possible by the compactness of the RD schemes. As a consequence of the first-order re-formulation of the equation, the number of linear relaxations performed at every Newton iteration was shown to increase only linearly with the grid size, not quadratically as typical for diffusion problems. The efficiency of the developed second-order schemes was demonstrated for linear and nonlinear advection–diffusion problems on highly refined grids, up to 30,000 nodes.

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In this paper, we propose a very simple extension of the second-order schemes to higher-order. We show that high-order spatial accuracy can be achieved simply by modifying the source term discretization. There are two approaches to the source term discretization: (1) reformulation of the source terms with their divergence forms and (2) correction to the trapezoidal rule for the source term discretization. The former technique is based on the divergence formulation of source terms proposed in Ref. [8]: write the source term in the divergence form and discretize it in the same way as the flux divergence term. The latter is based on a high-order correction to the trapezoidal rule, and thus called here the generalized trapezoidal rule. In either case, high-order accuracy is achieved by making low-order truncation error terms proportional to the residual, which thus vanish in the steady state and yield high-order accuracy. We solve the resulting implicit-residual equations by an implicit solver based on the second-order Jacobian matrix developed in the previous work [6]. As we will show, the implicit solver is as powerful as Newton's method; e.g., eight orders of magnitude reduction can be achieved in 10 iterations. To enable time-accurate computations, we employ high-order versions of the backward difference formulas (BDF), which are incorporated as source terms, and solve the implicit-residual equations by the implicit solver over each physical time step. In this manner, the steady state is made equivalent to the next physical time with all the benefits of the hyperbolic method retained. We note that the choice of the implicit time stepping method is independent of the developed high-order RD schemes, and thus other methods such as implicit Runge–Kutta methods or space-time methods can also be employed.

The high-order RD schemes developed in this work are significantly different from other high-order RD schemes in that our schemes are based on the first-order hyperbolic system formulation of the advection–diffusion equation [2]. In this approach, the loss of high-order accuracy in the intermediate Reynolds number, as discussed in Refs. [9–11], cannot occur because the advective and diffusive terms are fully integrated into a single hyperbolic system. If the original advection–diffusion equation is discretized, a high-order RD scheme needs to be developed for the diffusion term (i.e., second derivative) and then carefully combined with an advection scheme, e.g. by using a blending parameter as described in Ref. [10], to avoid the loss of accuracy. Furthermore, while high-order RD schemes based on high-order elements require extra degrees of freedom for each variable, our schemes are based on linear elements for any order of accuracy but require extra gradient variable to be added to the solution vector. Note that the number of extra variables in the high-order elements increase for higher-order accuracy, but the number of extra variables required in our approach is fixed and independent of the order of accuracy. Our approach is somewhat similar to those in Refs. [12–14], but again is significantly different by the use of first-order hyperbolic system formulation of the advection–diffusion equation and by the source term discretization techniques. It is emphasized that our schemes require only the first and second derivatives of the source term, or in some cases the first derivatives only; they do not require the gradient computation for the solution variables.

The third-order schemes developed in this paper are similar to the third-order finite-volume scheme of Katz and Sankaran [15,16] in that the second-order truncation error is eliminated by making it proportional to the residual and the upgrade is achieved by second-order accurate gradients. However, as we demonstrate in this paper, the proposed high-order RD schemes have several superior features: (1) implicit solver can be constructed by the Jacobian derived from a compact second-order RD scheme, (2) gradient computations are required for the source terms only, and not for the solution,(3) stiffness due to the second derivative of the diffusion term is completely eliminated, (4) higher-order schemes can be constructed beyond third-order (in extending it to multi-dimensions), and (5) the same order of accuracy is achieved for the gradients, as well. In particular, the fourth-order scheme constructed in Section 5 is remarkably more efficient because it does not require second derivatives of the source term, which are required in the schemes described in Refs. [15,16].

In this paper, we focus on one-dimensional linear and nonlinear advection–diffusion problems. It certainly serves as a basis for the development of high-order multi-dimensional RD schemes for more complex equations. Yet, more importantly, the one-dimensional high-order schemes developed in this paper could potentially bring significant improvements to practical problems such as material thermal response calculations of thermal protection systems of atmospheric entry vehicles [17–19], and the experimental aeroheating data reduction [20,21], which are based on one-dimensional analyses and routinely used in industries (e.g. NASA). The extension to higher dimensions is beyond the scope of the paper; it will be addressed in a subsequent paper.

The paper is organized as follows. In the next section, the time-dependent hyperbolic advection–diffusion system is described. In Section 3, a compact second-order residual-distribution scheme, a steady solver, and the second-order discretization are discussed. In Section 4, the third- and fourth-order RD schemes with source term reformulation are proposed. In Section 5, the third-, fourth, and sixth-order RD schemes with source term discretization are developed and proposed. Numerical results are then presented in Section 6. Finally, Section 7 concludes the study with remarks.

2. Time-dependent hyperbolic advection–diffusion system

We start with a linear advection–diffusion equation to simplify the discussion. We will extend the discussion later to a more general nonlinear advection–diffusion equation.

Consider the one-dimensional (1-D) time-dependent advection–diffusion equation:

$$\partial_t u + a\partial_x u = \nu\partial_{xx} u + \widetilde{S}(x), \tag{1}$$

where a and ν are both taken to be positive constant, and $\widetilde{S}$ is the source term. We will follow the procedure we described in Ref. [6] and rewrite the above equation as a first-order hyperbolic advection–diffusion system:

$$\partial_\tau u = -a\partial_x u + \nu\partial_x p - \frac{\alpha}{\Delta t}u + S(x), \tag{2}$$

$$\partial_\tau p = (\partial_x u - p)/T_r, \tag{3}$$

where the relaxation time, $T_r > 0$, is arbitrary and defined as described in Ref. [6], and S includes any existing source terms from the advection–diffusion problem, $\widetilde{S}$, as well as any additional terms that arise from the implicit time-stepping scheme, Δt is the physical time steps, and τ is the pseudo time step. Note that the $\partial_t p$ is taken as pseudo time derivative, $\partial_\tau p$.

The variable α depends on the order of the Backward-Differencing-Formula (BDF): 1 for the 1st-order (BDF1), 3/2 for the second-order (BDF2), 11/6 for the third-order (BDF3), 25/12 for the fourth-order, and 147/60 for the sixth-order time discretizations (see Table 1). The remaining terms in the BDF are stored in the source term function $S(x)$. It is well known that the BDF2 is *A-stable* and higher-order BDFs are not. Therefore, the second-order scheme is unconditionally stable, but higher-order BDFs are conditionally stable. Consequently, the stability of the higher-order schemes depends on the spatial discretization. Estimates for the maximum-allowable CFL numbers are given in Appendix A for a set of representative high-order schemes developed in this paper.

Towards the pseudo steady state, the variable p approaches the solution gradient and hence the above equation becomes identical

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