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Isogeometric Analysis and error estimates for high order partial differential equations in fluid dynamics

Anna Tagliabue [a,*], Luca Dedè [b], Alfio Quarteroni [a,b]

[a] MOX – Modeling and Scientific Computing, Mathematics Department "F. Brioschi", Politecnico di Milano, via Bonardi 9, Milano 20133, Italy
[b] CMCS – Chair of Modeling and Scientific Computing, MATHICSE – Mathematics Institute of Computational Science and Engineering, EPFL – École Polytechnique Fédérale de Lausanne, Station 8, Lausanne CH-1015, Switzerland

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ABSTRACT

In this paper, we consider the numerical approximation of high order Partial Differential Equations (PDEs) by means of NURBS-based Isogeometric Analysis (IGA) in the framework of the Galerkin method, for which global smooth basis functions with degree of continuity higher than $\mathcal{C}^0$ can be used. We derive a priori error estimates for high order elliptic PDEs under h-refinement, by extending existing results for second order PDEs approximated with IGA and specifically addressing the case of errors in lower order norms. We present some numerical results which both validate the proposed error estimates and highlight the accuracy of IGA. Then, we apply NURBS-based IGA to solve the fourth order stream function formulation of the Navier–Stokes equations for which we derive and numerically validate a priori error estimates under h-refinement. We solve the benchmark lid-driven cavity problem for Reynolds numbers up to 5000, by considering both the classical square cavity and the semi-circular cavity, which is exactly represented by NURBS.

1. Introduction

The numerical approximation of high order Partial Differential Equations (PDEs) represents a challenging task for the classical Galerkin Finite Element methods due to the need to use trial and test functions featuring high degree of continuity. This issue has been addressed by adapting existing Finite Element schemes or developing new numerical schemes. Specifically, the Discontinuous Galerkin (DG) [51] and local Discontinuous Galerkin (LDG) methods, firstly introduced in [18], were developed and adapted for solving high order PDEs; see e.g. [44] for fourth order PDEs and the references therein for more general cases. Analogously, non-conforming discretizations have been used to achieve the needed global regularity [15]; additionally, ad hoc techniques as continuous/discontinuous finite element approximations for fourth order PDEs have been developed in [25] in order to overcome the issue of defining $\mathcal{C}^1$-continuous basis for arbitrary shaped elements in dimensions greater than one. Currently, the golden standard in the framework of the standard Galerkin method with Lagrangian basis functions, consists in resorting to mixed formulations [28]. Spectral or pseudo-spectral domain decomposition techniques have also been used to approximate fourth order PDEs resulting from the Navier–Stokes equations in stream function formulation, see [46].

Isogeometric Analysis (IGA) is a recently developed computational methodology initiated with the work of Hughes et al. in [19] aiming at closing the existing gap between Computed Aided Design (CAD) and Finite Element Analysis (FEA). Based on the isogeometric paradigm, for which the same basis functions used to represent the known geometry are then used to approximate the unknown solution of the PDEs, IGA has been successfully used for the numerical approximation of a wide range of problems providing accurate and efficient solutions. An extensive discussion on the solution of both linear and nonlinear equations governing elasticity or fluid dynamics problems by means of IGA is provided in [20]. Moreover, IGA provides advantages in the numerical approximation of high order PDEs within the framework of the standard Galerkin formulation, since in IGA globally smooth basis functions can be eventually used. In particular, we refer to NURBS-based IGA, due to the large use of NURBS (Non-Uniform Rational B-Splines) [47] within the CAD technology, and above all, for the mathematical properties of these basis functions. We observe that, besides the possibility of offering simplified refinements procedures, NURBS allow to exactly represent some common geometries in engineering design, e.g. conic sections.

One of the major features of NURBS, which allows efficient numerical approximations of high order PDEs in the framework

* Corresponding author. Tel.: +39 02 2399 4604; fax: +39 02 2399 4586.
E-mail address: anna.tagliabue@polimi.it (A. Tagliabue).

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of the Galerkin method, consists in the fact that NURBS basis functions can be globally $\mathcal{C}^k$-continuous in the computational domain, with $k \geqslant 0$. This property allows a direct discretization of the weak form of the problem without the need to resort to mixed formulations, as typically is the case of FEA. In this respect, NURBS-based IGA has already been successfully used to solve high order PDEs. In [35,41] the fourth order Cahn–Hillard equations have been solved, while in [9,22,37] high order phase field models have been used for fracture modeling, topology optimization, and crystal growth, respectively. Similarly, in the field of Fluid Dynamics problems, the isothermal Navier–Stokes–Korteweg equations, which model isothermal vapor-liquid phase transitions, have been solved e.g. in [36]. In [6,7] structural problems for shell and plates have been solved with IGA, specifically Kirchhoff–Love models. In [2] a stream function, high order formulation has been used to solve planar elastic problems within the IGA framework, for which an estimation of the convergence rates of the errors with respect to the mesh size has been performed numerically. However, despite a significant numerical evidence, a complete, theoretical error analysis for high order PDEs has not been performed yet, especially for errors in lower order norms.

In this work, we provide a priori error estimates under h-refinement for the NURBS-based IGA approximation of high order scalar elliptic PDEs, extending the results presented in [3] for second order PDEs. Specifically, we review some approximation results presented in [3] and we focus on the derivation of the errors in lower order norms by means of the standard Aubin–Nitsche's arguments [45,54] for linear high order scalar elliptic PDEs. The derivation of the error estimates in lower order norms follows an approach similar to that of [54], for which the authors mention the possibility of using Splines as basis functions; in this respect, in the current work, we extend this concept to NURBS-based IGA. We highlight the dependence of the convergence rates on the order of the spatial differential operators, the regularity of the solutions, and the degree of the basis functions used. The convergence rates of the approximation errors of the IGA-Galerkin method with respect to the global mesh size h are verified by means of numerical tests with fourth and sixth order scalar PDEs.

As application, we consider the numerical approximation of the Navier–Stokes equations in stream function formulation [50], for which the incompressibility condition is fulfilled exactly in the computational domain; in this case, the standard Galerkin formulation yields a stable problem by construction. We derive a priori error estimates, including those in lower order norms, under h-refinement for the steady Navier–Stokes equations in stream function formulation, which represents a fourth order nonlinear elliptic PDE. Similarly to the linear case, we validate the error estimates by means of numerical tests. Then, we present a numerical study by means of NURBS-based IGA for the benchmark lid-driven cavity problem by comparing the results, up to Reynolds number 5000, with those available in literature. Namely, for the lid-cavity problem in a square, we refer to [10], which considers a spectral Chebychev collocation method, [26,27], where divergence-conforming B-Splines discretizations in the framework of IGA are developed, and [29], where a multigrid technique applied to finite difference approximations of the vorticity-stream function formulation of the Navier–Stokes equations is used. Moreover, by taking the advantage of the exact representation of conic sections allowed by NURBS, we present the numerical results for the lid-driven cavity problem in a semi-circular domain, a configuration that is considered e.g. in [14,16,33,43].

The outline of this work is as follows. In Section 2, we recall the basic notions of NURBS-based IGA in the framework of the Galerkin method, specifically for high order scalar elliptic PDEs. In Section 3, we derive the a priori error estimates for linear elliptic problems. In Section 4, we present, in view of the numerical tests, some high order PDEs and discuss the numerical approximation schemes. In particular, we present the stream function formulation of the Navier–Stokes equations and we derive a priori error estimates for the steady case. Finally, in Section 5, we report and discuss the numerical results. Conclusions follow.

2. NURBS-based Isogeometric Analysis

In this section, we recall the basic concepts of the B-Splines and NURBS basis functions and geometrical representation. Then, in Section 2.2, we briefly describe NURBS-based IGA in the framework of the Galerkin method for the solution of high order PDEs. For an extensive overview of B-Splines and NURBS, see for instance [20,47]; for more details related to NURBS-based IGA, we refer the interested reader to e.g. [19–21]. The notation used in this work is similar to the one used in [3,5].

2.1. B-Splines and NURBS

A *knot vector* is a set of non-decreasing real numbers, representing coordinates in the *parameter space*. We indicate the knot vector as $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$, where ξ_i is the i-th knot, with the knot index $i \in \{1, \dots, n+p+1\}$ characterized by the polynomial degree p and the number of basis functions n defining the B-Splines basis, respectively. By convention, we assume that $\xi_1 = 0$ and $\xi_{n+p+1} = 1$, such that the parametric domain is defined as $\widehat{\Omega} := (\xi_1, \xi_{n+p+1}) = (0,1) \subset \mathbb{R}$. Knots may be repeated with the number of repetitions indicating its *multiplicity*. A knot vector is said to be *open* if its first and last knots appear $p+1$ times; specifically, in this work we consider this case. In order to introduce the concept of *mesh* elements in the parametric domain, we collect all the r distinct and ordered knots of Ξ, say ζ_j for $j = 1, \dots, r$, into a vector $\mathcal{Z} = \{\zeta_1, \dots, \zeta_r\}$, with $\zeta_1 \equiv \xi_1 = 0$ and $\zeta_r \equiv \xi_{n+p+1} = 1$. In particular, the one dimensional *mesh* over $\widehat{\Omega}$, say $\mathcal{Q}_h$, is defined as the collection of the subdomains bounded by two distinct knots, i.e.:

$$\mathcal{Q}_h := \{Q = (\zeta_j, \zeta_{j+1}) \ : \ j = 1, \dots, r-1\}; \tag{2.1}$$

we indicate with $\hat{h} := \max\{\hat{h}_Q : \ Q \in \mathcal{Q}_h\}$ the *global mesh size* in the parametric domain $\widehat{\Omega}$, where $\hat{h}_Q := \mathrm{diam}(Q)$ for all $Q \in \mathcal{Q}_h$. Moreover, since the multiplicity of the knots has important implications in the regularity properties of the basis functions, an auxiliary vector is defined in relation with $\mathcal{Z}$; specifically, we introduce the vector $\mathcal{M} := \{m_1, \dots, m_r\}$, with $m_j \geqslant 1$ representing the multiplicity of the knot value ζ_j, for $j = 1, \dots, r$.

By means of the *Cox–de Boor recursion formula* [20,47], univariate B-Splines basis functions $N_i : \widehat{\Omega} \to \mathbb{R}$ for $i = 1, \dots, n$, are built as piecewise polynomials of degree p with compact support over the interval (ξ_i, ξ_{i+p+1}). The basis functions are everywhere pointwise non-negative and $\mathcal{C}^\infty$-continuous, except in the knot values ζ_j, where they are only $\mathcal{C}^{p-m_j}$-continuous. In particular, we define for all $j = 1, \dots, r$, the smoothness integer parameters $k_j = p - m_j + 1$ such that $0 \leqslant k_j \leqslant p$, we collect them in a vector $\mathcal{K} = \{k_1, \dots, k_r\}$, and we introduce the minimum integer parameter $k_{\min} := \min_{j=2,\dots,r-1}\{k_j\}$. We observe that, according to the definition of the vector $\mathcal{K}$, in the knot ζ_j, for $j = 1, \dots, r$, the basis functions are $\mathcal{C}^{k_j - 1}$-continuous.[1] The B-Splines space built from the basis functions in the parametric domain $\widehat{\Omega}$ reads:

$$\mathcal{S}_h := \mathrm{span}\{N_i\}_{i=1}^n. \tag{2.2}$$

By definition, the B-Splines in $\mathcal{S}_h$ are globally $\mathcal{C}^{k_{\min}-1}$-continuous. An example of B-Splines basis functions of degree $p = 2$, exhibiting different regularities across the knots, is provided in Fig. 1.

[1] We remark that in this paper, we consider the minimum smoothness parameter $k_{\min}$ to be larger than one unit with respect to the conventional notation of [19,20], since we use the same notation of [3] for deriving a priori error estimates.