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A novel upwind-based local radial basis function differential quadrature method for convection-dominated flows

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ABSTRACT

In this paper, a new upwind technique for local radial basis function differential quadrature (LRBF-DQ) method is proposed to solve the convection-dominated flow problems. By using a modified Euclidean distance function according to the local flow direction and the value of parameter that controls the convection effect, the local support in the formulation of LRBF-DQ can be chosen in a way shifting towards the upstream direction to form a comet-like shape. The upwind effect is therefore naturally incorporated when computing the weighting coefficients for LRBF-DQ method. The capability of the proposed method is examined by solving two-dimensional convection–diffusion equation with various Peclet numbers and magnetohydrodynamics (MHD) problems with very high Hartmann numbers. The results show that remarkable improvement of accuracy can be achieved by the current upwind-based LRBF-DQ method than the conventional ones.

1. Introduction

Convection is a common mechanism occurring in the flow problems of various fluids. In many interdisciplinary science and engineering applications, the problems are dominated by considerable convection effect, which are very challenging in the point of view of numerical simulation since dense grid resulting large computational cost is needed to eliminate the convection effect on the grid scale. Spurious oscillations may occur with insufficient computational nodes or unsuitable numerical schemes [1]. The development of efficient, reliable, and accurate numerical methods for the simulation of convection-dominated flows is therefore a crucial subject when studying the relevant problems. Considerable works of upwind schemes for finite difference method (FDM) and finite volume method (FVM) had been developed for the numerical solutions of convection-dominated flows, such as first order upwind difference (FOU) [2], second order upwind difference (SOU) [3], and quadratic upstream interpolation for convective kinetics (QUICK) [4] to name a few. On the other hand, stabilized finite element method (FEM) using modified weighting functions in the Petrov–Galerkin formulations such as the streamline upwind Petrov–Galerkin method (SUPG) [5] and the application of hierarchical basis functions [6] were developed to study the convecting flows. However, the accuracy and stability of the aforementioned mesh type numerical methods rely on good mesh quality. The generation of mesh with good quality is in general not a trivial issue, especially for complex boundary geometry in three dimensions. Development of meshless numerical methods to the numerical solutions of partial differential equations thus attracted considerable interests over the past decades [7–9].

In this paper, the LRBF-DQ method introduced by Shu et al. [10], which is a meshless numerical method and can be easily implemented to solve problems in arbitrary-shaped domain, is adopted to solve the convection-dominated flow problems. In the formulation of LRBF-DQ, the radial basis functions (RBFs) [11] are employed as the basis functions to compute the weighting coefficients for the differential operators, which extends the idea of original differential quadrature (DQ) method proposed by Bellman and Casti [12] and Bellman et al. [13] that can only be applied to problems in Cartesian grid. LRBF-DQ has been applied to solve inviscid compressible flows [14], natural convection [15], three-dimensional incompressible viscous flows [16,17], long waves in shallow water [18], the vibration analysis of membranes [19] and heat conduction problems [20,21], which show the capability of LRBF-DQ for solving various kinds of engineering problems.

In conventional LRBF-DQ proposed by Shu et al. [10], the local support of a computational node is chosen as a circular domain centered at the reference node. The weighting coefficients of differential operators computed by the neighbor nodes within the local support thus cannot properly distinguish the influence from upstream or downstream. In the current paper, the local support of a reference node is chosen in a way shifting towards the upstream direction to form a comet-like shape, whose concept is similar to

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the upwind schemes developed in FDM and FVM. This can be done by modifying the Euclidean distance function according to the local flow direction and the value of parameter that controls the convection effect when sorting the nearest neighbor nodes. The resulted shape of shifted local support thus depends on the strength and direction of convecting flow and exhibits the upwind effect naturally. It is mentioned in [14], an upwind LRBF-DQ method is proposed to simulate inviscid compressible flows by evaluating the flux at the mid-point between the reference node and its supporting node through approximate Riemann solvers.

The proposed upwind-based LRBF-DQ is validated by solving convection–diffusion equation with various Peclet numbers and MHD problems with very high Hartmann numbers in two dimensions. MHD flow problems have been solved by many numerical methods such as FEM [22,23], element-free Galerkin (EFG) method [24,25], exponential high-order compact (EHOC) difference scheme [26], meshfree point collocation method [27], and DQ method [28,29] with Hartmann number varying from 5 to 10^6. The Hartmann number in MHD flow problems is similar to the Peclet number in convection–diffusion equation, which controls the strength of the convection effect. Therefore, MHD problems are suitable to examine the capability of the numerical method when handling strong convecting flows.

The organization of the paper follows. Brief introduction to the formulation of LRBF-DQ as well as the proposed upwind scheme are presented in the next section. The governing equations of the problems are given in Section 3. In Section 4, the current upwind-based LRBF-DQ method is applied to solve two-dimensional convection–diffusion and MHD flow problems with various Peclet or Hartmann numbers, respectively. The results are compared with those solved by other methods or analytical solutions. Finally, remarks and discussions are drawn in Section 5.

2. Numerical method

2.1. Formulation of LRBF-DQ method

In this section, a brief introduction to the formulation of LRBF-DQ method is given. For detailed descriptions, the reader is referred to [10]. The essential idea of LRBF-DQ method is to approximate the function derivatives at a reference node by the linear weighted sum of the function values at neighbor nodes within the local support domain, which is an extension of the DQ method by Bellman and Casti that approximates the derivatives by the function values at all nodes in the global domain. The mth order derivative of a function $f(\mathbf{x})$ with respect to x at $\mathbf{x}=\mathbf{x}_i$ for LRBF-DQ method can be expressed as

$$f_x^{(m)}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_i} = \sum_{j=1}^{N_L} w_{ij}^{(mx)} f(\mathbf{x}_j), \quad \text{for } i = 1,\ldots,N, \tag{1}$$

where N_L is the number of nodes within local support of reference node $\mathbf{x}_i$, N is the number of global nodes, and $w_{ij}^{(mx)}$ are the weighting coefficients for mth order derivative along x-direction. In the present LRBF-DQ, the unknown function $f(\mathbf{x})$ is approximated by the linear combination of the multiquadrics (MQs) [30], since it is the most accurate one among various RBF-based interpolation methods [31]. The MQ function centered at $\mathbf{x}_j$ can be expressed as

$$\phi_j(\mathbf{x}) = \sqrt{|\mathbf{x}-\mathbf{x}_j|^2 + c^2}, \quad c > 0, \tag{2}$$

with c denoting a shape parameter. According to the principle of superposition, all the basis functions should satisfy the relation described in Eq. (1) and can be expressed in matrix form as

$$\frac{\partial^{(m)}}{\partial x^{(m)}}\begin{bmatrix}\phi_1(\mathbf{x}_i)\\ \phi_2(\mathbf{x}_i)\\ \vdots \\ \phi_{N_L}(\mathbf{x}_i)\end{bmatrix} = \begin{bmatrix}\phi_1(\mathbf{x}_1) & \phi_1(\mathbf{x}_2) & \cdots & \phi_1(\mathbf{x}_{N_L})\\ \phi_2(\mathbf{x}_1) & \phi_2(\mathbf{x}_2) & \cdots & \phi_2(\mathbf{x}_{N_L})\\ \vdots & \vdots & \ddots & \vdots\\ \phi_{N_L}(\mathbf{x}_1) & \phi_{N_L}(\mathbf{x}_2) & \cdots & \phi_{N_L}(\mathbf{x}_{N_L})\end{bmatrix}\begin{bmatrix}w_{i1}^{(mx)}\\ w_{i2}^{(mx)}\\ \vdots\\ w_{iN_L}^{(mx)}\end{bmatrix}, \tag{3}$$

for $i=1,\ldots,N$. Therefore the weighting coefficients for the mth order derivative along x direction of the function $f(\mathbf{x})$ at point $\mathbf{x}_i$ can be obtained by solving the above linear equations. Similarly, the weightings for the derivatives along other directions can be obtained. To satisfy the zeroth-order consistency condition, a constant function as an additional basis function is considered, which results in a condition $\sum_{j=1}^{N_L} w_{ij} = 0$ and is used to determine a suitable shape parameter. In practice, the weightings are computed using different c in a feasible range. The one with the minimum residual will be chosen and is expected to satisfy the zeroth-order consistency condition as possible.

2.2. Upwind scheme for LRBF-DQ method

Now we turn the attention to the proposed novel upwind scheme for LRBF-DQ method. In the original LRBF-DQ method [10], the local support of a reference point is chosen as a circular domain centered at that point. A practical way is to choose the N_L nearest neighbor points as the supporting nodes. The computed weighting coefficients cannot distinguish the influence from upstream or downstream. When the downstream boundary condition exhibits a severe change, strong convection effect may result spurious oscillations.

Here we adopt an idea similar to that developed in FDM or FVM, the local support of a reference node is shifted towards the upstream direction and its shape is modified to a comet-like geometry. The modification of the geometry should satisfy physical parameter that controls the strength of convection. This can be done by using a modified Euclidean distance function defined as

$$R^* = \sqrt{(R\sin\theta)^2 + (w_R R\cos\theta)^2}, \tag{4}$$

where $R = |\mathbf{x}_i - \mathbf{x}|$ is the Euclidean distance between $\mathbf{x}$ and the reference node $\mathbf{x}_i$, θ is the angle between the flow velocity $\mathbf{u}$ and $\mathbf{r} = \mathbf{x}_i - \mathbf{x}$ as depicted in Fig. 1, and w_R is a weighting factor denoted by

$$w_R = (\log F)^{-\mathrm{sgn}(\mathbf{r}\cdot\mathbf{u})} \tag{5}$$

with F denoting the physical parameter controlling the convection strength and the sign function defined as

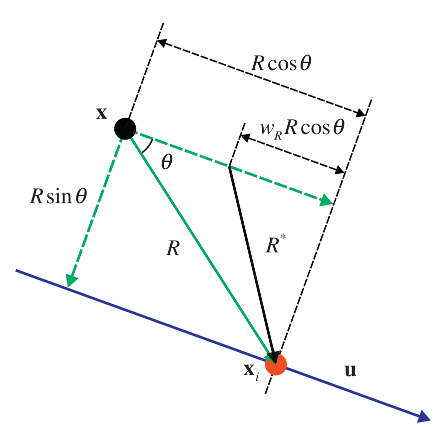

Fig. 1. Schematic sketch of the modified Euclidean distance function.