Computers & Fluids 88 (2013) 440–451

Contents lists available at ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

Diffusion in inhomogeneous flows: Unique equilibrium state in an internal flow

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ARTICLE INFO

Article history:
Received 2 November 2012
Received in revised form 10 July 2013
Accepted 2 October 2013
Available online 12 October 2013

Keywords:
Navier–Stokes equation
Inhomogeneous flow
Enstrophy transport equation
Direct numerical simulation
Diffusion
Rectangular lid driven cavity

ABSTRACT

The role of diffusion in creating rotationality (enstrophy) is studied here and a transport equation for enstrophy is derived to explain this connection. As an illustration, flow instabilities and pattern formation are investigated here for an inhomogeneous internal flow with definitive boundary conditions. Results obtained by direct numerical simulation (DNS) of flow inside a two-dimensional rectangular lid driven cavity (RLDC) show that diffusion is responsible in forming patterns at a post-critical Reynolds numbers. The transport equation for enstrophy derived from the Navier–Stokes equation in Eulerian framework helps to explain the enstrophy spectrum in flows, specially in 2D flows, where vortex stretching is absent as the dominant energy cascade mechanism to small scales. For the 2D flow in RLDC, diffusion and convection provide a unique equilibrium state in an intermediate post-critical range of Reynolds number around 6000. This is independent of the geometric aspect ratio (height to width of the cavity) of the cavity greater than or equal to two. Such equilibrium can be observed in numerical simulations, only when special care is exercised for diffusion discretization at high wavenumbers. Another motivation in this work is to show that diffusion and dissipation are not identical for inhomogeneous flows, as opposed to equating these in studies of homogeneous turbulent flows. Organized enstrophy is shown as a consequence of over-riding action of diffusion in creating rotationality in this flow.

1. Introduction

Investigation on the true role of diffusion has remained a problem, ever since the time when its role was considered as stabilizing fluid flow by damping disturbances, attributed to Kelvin, Helmholtz and Rayleigh [1]. Equating viscous diffusion with dissipation was the sole reason for early instability studies to ignore diffusion, as discussed in [1,2]. However, such studies were unable to explain instability of flow over a flat plate, while the same flow was successfully investigated by solving Orr-Sommerfield equation (OSE) [3–5], which includes viscous diffusion in the formulation. It was thought that retaining diffusion is equivalent to producing an appropriate phase shift for a positive feedback, which leads to flow instability.

Doering and Gibbon [6] studied the enstrophy transport for two-dimensional periodic flows and obtained the evolution of integrated enstrophy over the full domain as

$$\frac{d}{dt}\left(\frac{1}{2}\|\omega\|_2^2\right) = -\nu\|\nabla\omega\|_2^2 \qquad (1)$$

where ω is the vorticity and ν is the kinematic viscosity. Here, the enstrophy is defined over the full periodic domain by $\|\omega\|_2^2$. Thus, one notes the effects of diffusion as strictly dissipative for periodic flows viewed globally. In performing DNS of flows, one discretizes all the terms and obtains the numerical solution without any ambiguity. However, the point of view of equating diffusion with dissipation is often used, as given above in Eq. (1), while interpreting DNS results of homogeneous turbulent flows [7]. However if diffusion is viewed instantaneously at any point in a flow, then the effects of diffusion is not strictly dissipative, as will be explained here. When one looks at the time-averaged kinetic energy of turbulent flows globally, effects of diffusion is again seen to be as dissipative [8,9]. As shown in Eq. (4.34) of [9], time-average of the diffusion term of Navier–Stokes equation manifests itself as a combination of (i) a strictly dissipation term and (ii) another *viscous transfer* term. However, *the viscous transfer term integrates to zero over the whole flow by the divergence theorem. This term is sometimes also referred to as diffusive, because it is zero for homogeneous turbulence*. The authors furthermore add that *the viscous transfer term is negligible at high Reynolds numbers, except within the thin viscous layers very near any solid surfaces* while on the other hand, *the dissipative term is of crucial importance to turbulence energetics everywhere*. Similar observations are made in Section 3.3 of [8], with respect to time-averaged turbulent kinetic energy. In the present

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http://dx.doi.org/10.1016/j.compfluid.2013.10.005

investigation, we look at the instantaneous local behavior of diffusion term and demonstrate in a flow the existence of a unique equilibrium state where the diffusive nature dominates over the dissipative nature of the viscous term in Navier–Stokes equation.

We also show by an appropriate analysis requiring the consideration of the time-accurate total mechanical energy (as opposed to time-averaged property of only the kinetic energy) of the flow for which the action of viscous terms is not directly apparent. If one constructs an equation for the total mechanical energy, as suggested in [10] and developed in [11], then the role of diffusion becomes clearer, as described in the following. One writes the Navier–Stokes equation in rotational form for this analysis as

$$\frac{\partial \vec{V}}{\partial t} - \vec{V} \times \vec{\omega} = -\nabla\left(\frac{p}{\rho} + \frac{\vec{V}\cdot\vec{V}}{2}\right) + \nu\nabla \times \vec{\omega} \tag{2}$$

where different variables represent their usual meanings and the viscous effects is via the last term on the right hand side, written as the curl of the vorticity vector, multiplied by the kinematic viscosity.

Describing the total mechanical energy (E) by

$$E = \frac{p}{\rho} + \frac{\vec{V}\cdot\vec{V}}{2}$$

and taking a divergence of the above Navier–Stokes equation yields the distribution of E by the following equation

$$\nabla^2 E = \nabla \cdot (\vec{V} \times \vec{\omega}) \tag{3}$$

Note that the viscous term drops out identically due to a vector identity and the right hand side originate strictly from convection term. However, the right hand side of the above equation can be expressed using the vector identity in further simplification of this equation

$$\nabla \cdot (\vec{V} \times \vec{\omega}) = \vec{\omega}\cdot\vec{\omega} - \vec{V}\cdot(\nabla \times \omega)$$

Denoting the instantaneous point property of enstrophy by $\Omega_1 = \vec{\omega}\cdot\vec{\omega}$, Eq. (3) can be written as

$$\nabla^2 E = \Omega_1 - \vec{V}\cdot(\nabla \times \vec{\omega}) \tag{4}$$

This equation shows the relevance of enstrophy and the diffusion operator to be central in distributing total mechanical energy. In [7], a similar equation has been written for the static pressure (see Eq. (1.2) of the reference) which in present notations is given by

$$\nabla^2\left(\frac{p}{\rho}\right) = (\Omega_1 - \epsilon/\nu)/2 \tag{5}$$

where $\epsilon = 2\nu s_{ij}s_{ij}$ and s_{ij} is the symmetric part of the strain tensor. This equation is wrongly stated to be valid only for homogeneous turbulence. Eq. (4) is written for any general flow derived from Navier–Stokes equation without making any assumption or simplification. One notes that the term on the right hand side of Eq. (4) can be written as $\vec{V}\cdot\frac{\nabla^2\vec{V}}{\nu}$, in drawing an analogy with the term ϵ/ν, on the right hand side of Eq. (5), even though the right hand side of Eq. (4) purely originates from convection term. This source of confusion prompted the authors in [7,12,13], to equate the roles of enstrophy and dissipation. One of the motivations here is to highlight the connection between diffusion and enstrophy for flows. The development and use of total mechanical energy equation to study any flow instability is described in detail in [1,11].

In trying to understand the role of diffusion in creating rotationality, an evolution equation is also developed here for enstrophy, as a point property and its higher powers for any flow. This exercise explains the roles of diffusion, dissipation and creation of rotationality progressively to smaller scales. To demonstrate that this is valid for any flow, we focus on a 2D flow, which does not have the presence of vortex stretching to create smaller scales.

Reported DNS in [7], used Fourier spectral discretization in space and second order Runge–Kutta time integration to solve Navier–Stokes equation. This space–time dependent discretization is very restrictive in parameter space, due to its numerical instability and also due to its high dispersion error, as shown by spectral analysis in the appendix using the 1D convection equation. It is obvious that any method which cannot solve this simple convection equation, is practically of little use in solving more complex Navier–Stokes equation. The dynamical equilibrium in flows is a balance between convection and diffusion processes, both of which have to be captured correctly in equal measure. One of the salient features of the presented results here is to show the existence of a universal equilibrium between convection and diffusion in Navier–Stokes equation. This can be captured only by carefully designed numerical methods explained in the next section and appendix.

There have been significant progresses made in developing high accuracy compact schemes, which are dispersion relation preserving (DRP) and has been used for inhomogeneous flows. A similar method has been used in [14,15] to simulate an inhomogeneous zero pressure gradient boundary layer from the receptivity to a fully developed 2D turbulent stage, displaying k^{-3} spectrum for the energy. One of the motivations here is to show that for 2D flows, rotationality is created at different scales via the enstrophy cascade. This establishes a link between diffusion and enstrophy for a wall-bounded inhomogeneous flow.

Here, the flow inside a RLDC driven by uniform translation of the top lid (U_∞) is used as an example to reveal the role of diffusion in Navier–Stokes equation, where pronounced rotationality is created by simple translation of the top lid. It is well known [16] that turbulence is characterized by many attributes, out of which the primary ones being rotationality and broad-band energy spectrum created by various instability mechanisms.

Flow in a square LDC has been studied and a unique topology (triangular core vortex and gyrating satellite vortices) is described in [17,18]. This was obtained with the help of highly accurate discretization of convection and diffusion processes in the flow. Flow in RLDC is more complicated due to the presence of multiple cells having distinct vortical structures. The upper cell of RLDC resembles the flow in a square LDC, which in turn, drives the cell below and so on. The rotational flow structures seen in various cells of RLDC are caused by the translational motion of the lid, with each cell showing presence of vortices of both signs.

The manuscript is formatted in the following manner. In the next section, governing equations and the numerical methods to solve 2D flow inside the RLDC are described. This is followed by a section describing the flow inside RLDC, with respect to the instability sequence, topology and Hopf bifurcation of the flow. To explain this instability sequence and induced rotationality, transport equation for enstrophy has been derived in Section 4. In Section 5, we emphasize the requirements on diffusion discretization in DNS. This is followed by summary and conclusion of the results. In the appendix, the spectral analysis of numerical schemes used for convection equation has been carried out.

2. Governing equations and numerical formulation

We have used the streamfunction–vorticity (ψ, ω)-formulation of Navier–Stokes equation to obtain numerical results reported for the RLDC shown in Fig. 1. This formulation allows satisfaction of solenoidality for velocity and vorticity identically. The non-dimensional form of vorticity transport equation (VTE) for 2D flows is given by