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An unstructured finite volume method for large-scale shallow flows using the fourth-order Adams scheme

Abdelaziz Beljadid [a,*], Abdolmajid Mohammadian [a], Hazim M. Qiblawey [b]

[a] Department of Civil Engineering, University of Ottawa, 161 Louis Pasteur, Ottawa, Ontario K1N6N5, Canada
[b] Department of Chemical Engineering, Qatar University, P.O. Box 2713, Doha, Qatar

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ABSTRACT

In this paper, we introduce a new upwind finite volume method using unstructured grids for large-scale shallow flows. This method uses a high-order upwind scheme for the calculation of the numerical flux, and the fourth-order Adams method with a splitting approach for time integration. The process includes three stages: in the first and third steps the Coriolis term is integrated analytically, and in the second step the flux term is integrated numerically. Most upwind schemes perform well for gravity waves but they lead to a high level of damping or numerical oscillations for Rossby waves. The proposed method presents the advantage that it performs well for both gravity and Rossby waves. The use of fourth-order Adams method without any iteration on the corrector is enough to suppress the short-wave numerical noise without damping the long waves that are essential in the transport of energy Rossby waves, in large-scale oceanic and atmospheric flows.

1. Introduction

Shallow water equations (SWEs) are used to describe many physical phenomena in oceans, rivers, the atmosphere, etc. These equations are applicable when the vertical velocity component is negligible compared to the horizontal components, and are obtained by assuming hydrostatic pressure distribution (e.g., [29]). The three-dimensional incompressible Navier–Stokes equations are averaged over the depth to obtain the SWEs. In the absence of viscous terms, SWEs can be considered a hyperbolic system. The finite volume (FV) methods are most convenient for modeling these systems since they have a conservative form. Upwind finite volume (UFV) methods can numerically solve these systems with good accuracy and an acceptable computational cost.

UFV schemes use exact or approximate methods to solve the Riemann problem at the interface of computational cells. Godunov's method [11,10] is the most popular scheme using the exact solution of the Riemann problem. Its extension to second-order and to high-order schemes is given by Van Leer [27] and Colella and Woodward [7], respectively. The exact algorithms are computationally expensive compared to the approximate methods. Roe's method [22], which is applied in this work, is the most popular approximate method. It requires an accurate estimation of parameter values near the interface on both sides of the computational cell. In the presence of source terms in the SWEs, the UFV schemes may lead to numerical oscillations due to the imbalance between the source and flux terms. To overcome this problem, some special treatments can be applied for balancing the source and flux terms. A large number of studies have been conducted in this direction, such as Vázquez-Cendón [28], Gallouet et al. [9], Mohammadian et al. [21], Mohammadian and Le Roux [20], and Stewart et al. [25]. Other studies have been conducted to evaluate the performance of various schemes for large-scale shallow flows (e.g., [31,8,14,13,17,12,30,16]). Nevertheless, UFV methods are considered in a limited number of studies (e.g., [18,19,6,1]). The performance of numerical methods is greatly influenced by the temporal schemes used. Total Variation Diminution (TVD) temporal integration methods, developed by Shu and Osher [24], are among the most popular temporal integration schemes. They are widely used for their ability to avoid oscillations and to maintain stability. Furthermore, some higher-order TVD schemes are insensitive to the values of Courant–Friedrichs–Lewy (CFL) numbers and present highly accurate results over a wide range of CFL numbers [2].

Beljadid et al. [2] studied the performance of UFV schemes with TVD Runge–Kutta methods for temporal integration. Several aspects were examined, including mass and energy conservation, numerical diffusion, and numerical oscillations for Kelvin, Yanai, Poincaré, and gravity waves. The accuracy of various schemes was analyzed for different types of waves in order to identify the most accurate and efficient numerical schemes. Through numerical

* Corresponding author.
E-mail address: abelj016@uottawa.ca (A. Beljadid).

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experiments, it was demonstrated that a third-order TVD Runge–Kutta method (TVDRK3) combined with the upwind-centered scheme provides accurate results for Kelvin, Yanai, Poincaré, gravity, and inertia gravity waves. The TVDRK3 method with the upwind-centered scheme was found to be a good choice for these types of waves. It was shown that the results remained accurate for a wide range of CFL numbers, which is important in practical applications. Moreover, this scheme presents good stability properties even for large spatial variation of computational cells, usually present in unstructured grids. However, this method fails in the modeling of Rossby waves, which have a particular behavior and are difficult to capture by several well-known upwind schemes. In this paper we propose a new upwind finite volume method which presents a good improvement for the modeling of Rossby waves. A high-order spatial scheme based on polynomial fitting is proposed. Operator splitting and the fourth-order Adams method are used for temporal integration.

The paper is organized as follows: SWEs are presented in Section 2. In Section 3, the proposed finite volume method is described. Section 4 presents some numerical experiments for equatorial Rossby waves. In Section 5, some numerical experiments are performed using the proposed method for nonlinear SWEs. Some concluding remarks complete the study.

2. Shallow water equations

In this section, linear and nonlinear shallow water equations are presented. The conservative form of the 2D shallow water equations is written as [29]:

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}=\mathbf{S} \tag{1}$$

The linear and nonlinear equations are defined in terms of parameters **U**, **E**, **G** and **S**.

2.1. Linear SWEs

For linear shallow water equations, the parameters **U**, **E**, **G** and **S** are defined as:

$$\mathbf{U}=\begin{bmatrix}\eta\\ u\\ v\end{bmatrix},\qquad \mathbf{E}=\begin{bmatrix}Hu\\ g\eta\\ 0\end{bmatrix},\qquad \mathbf{G}=\begin{bmatrix}Hv\\ 0\\ g\eta\end{bmatrix},\qquad \mathbf{S}=\begin{bmatrix}0\\ fv\\ -fu\end{bmatrix}, \tag{2}$$

where η represents the water surface elevation, u and v are the depth-averaged velocity components in the x- and y-directions, respectively, f is the Coriolis parameter, g is the gravity acceleration, and $(H+\eta)$ is the total water depth.

The term **S** may include various source terms such as bed friction, bed topography, and wind stress. Since this paper concentrates on Rossby waves, the source term **S** is assumed to include the Coriolis parameter.

The beta-plane approximation to the Coriolis parameter is considered ($f=\beta y$), where β is the linear coefficient of variation of f with respect to y. The variable y is considered as the meridional distance from the equator (positive northward). The parameter β is given as:

$$\beta = 2\mathbf{\Omega}/\mathbf{R} = 2.29\times 10^{-11}\ \mathrm{m}^{-1}\mathrm{s}^{-1} \tag{3}$$

where $\mathbf{\Omega}$ and **R** are the angular speed of the Earth's rotation and the mean radius of the Earth, respectively ($\mathbf{\Omega}=7.29\times10^{-5}$ rad s^{-1}, **R** = 6371 km)

The dimensionless form of SWEs is used in this paper. The model Eqs. (1) and (2) are converted into dimensionless form on an equatorial beta-plane using the variables $\tilde{x}=x/L^*$, $\tilde{y}=y/L^*$, $\tilde{\eta}=\eta/H^*$, $\tilde{u}=u/U^*$ and $\tilde{v}=v/U^*$. The reference values of the depth (H^*), time (T^*), length (L^*) and velocity (U^*) scales are expressed as:

$$\begin{aligned}
H^* &= H\\
T^* &= \beta^{-1/2}(gH)^{-1/4}\\
L^* &= \frac{1}{\beta T^*}\\
U^* &= V^* = \frac{L^*}{T^*}
\end{aligned} \tag{4}$$

The resulting system, the Jacobian matrix, and the corresponding eigenvalues and eigenvectors are given in Appendix A.

2.2. Nonlinear SWEs

For nonlinear shallow water equations, the parameters **U**, **E**, **G** and **S** are defined as:

$$\mathbf{U}=\begin{bmatrix}h\\ hu\\ hv\end{bmatrix},\qquad \mathbf{E}=\begin{bmatrix}hu\\ hu^2+0.5gh^2\\ huv\end{bmatrix},\qquad \mathbf{G}=\begin{bmatrix}hv\\ huv\\ hv^2+0.5gh^2\end{bmatrix} \tag{5}$$

The source term **S** is assumed to include the Coriolis effect

$$\mathbf{S}=(0, fhv, -fhu)^t \tag{6}$$

where h is the total fluid depth.

In the presence of the Coriolis effect, the nonlinear SWEs are converted into a dimensionless form on an equatorial beta-plane using the variables $\tilde{x}=x/L^*$, $\tilde{y}=y/L^*$, $\tilde{h}=h/H^*$, $\tilde{u}=u/U^*$, and $\tilde{v}=v/U^*$. The characteristic time (T^*), length (L^*) and velocity (U^*) scales are expressed in terms of the parameter β in the same way using Eq. (4), where the parameter H^* is the mean water depth.

When the Coriolis force is absent, the following reference parameters are used to convert the nonlinear SWEs to a dimensionless form:

$$\begin{aligned}
T^* &= L^*\Big/\sqrt{gH^*}\\
U^* &= V^* = \frac{L^*}{T^*}
\end{aligned} \tag{7}$$

where the characteristic length L^* can be arbitrarily chosen and the parameter H^* can be chosen with the same order as the mean water depth h in the system.

3. Finite volume method

An upwind finite volume method on an unstructured grid is employed in this paper. The variables are located at the geometric centers of the computational grids. Each triangle represents a control volume. The SWEs are integrated over every control volume as:

$$\int_\Omega \left(\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}-\mathbf{S}\right)d\Omega = 0 \tag{8}$$

where Γ and Ω denote the boundary and the area of the domain, respectively.

By using the divergence theorem, the flux integral is transformed into a boundary integral:

$$\int_\Omega \left(\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}\right)d\Omega = \int_\Gamma \mathbf{F}\cdot\mathbf{n}\,d\Gamma \tag{9}$$

where $\mathbf{F}=(\mathbf{E},\mathbf{G})^t$ is the flux vector and **n** is the unit outward normal vector to the boundary Γ. Then, (8) leads to