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On the Darboux integrability of Blasius and Falkner–Skan equation

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ABSTRACT

We study the Darboux integrability of the celebrated Falkner–Skan equation $f''' + ff'' + \lambda(1 - f'^2) = 0$, where λ is a parameter. When $\lambda = 0$ this equation is known as Blasius equation. We show that both differential systems have no first integrals of Darboux type. Additionally we compute all the Darboux polynomials and all the exponential factors of these differential equations.

1. Introduction and statement of the main results

The *Falkner–Skan equation* is

$$f''' + ff'' + \lambda(1 - f'^2) = 0, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter. This equation was first derived in [6] as a model of the steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge. The special case $\lambda = 0$, in which the wedge reduces to a flat plate, is called *Blasius equation* and was considered for a first time in [2].

Both equations are the subject of an extensive literature. For the derivation of this equation see [1]. For the existence and uniqueness of the solutions see, for example, [19,22,5,18,3,13] and references therein. Recently there has been also a renewed interest in the mathematical aspects of the Falkner–Skan equation. The dynamic features of this equation such as the existence of oscillating and periodic orbits have been studied in [10–12,16]; and for more recent works on the bifurcations in this equation see [14,21,20].

In MathSciNet appears in this moment 214 articles related with the Falkner–Skan equation, but any of these papers analyze the integrability or non-integrability of this equation. In this work we are interested in the Darboux integrability of Blasius and Falkner–Skan equation. Before we state our main result (Theorem 1) we need to introduce some definitions and auxiliary results.

We can express (1) as a system of differential equations

$$\dot{x} = y, \quad \dot{y} = z \quad \dot{z} = -xz - \lambda(1 - y^2), \tag{2}$$

and the associated vector field is

$$\mathcal{X} = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + [-xz - \lambda(1 - y^2)]\frac{\partial}{\partial z}. \tag{3}$$

Let $U \subset \mathbb{R}^3$ be an open subset. We say that the non-constant function $H : U \to \mathbb{R}$ is the *first integral* of the polynomial vector field (3) on U associated to system (2), if $H(x(t), y(t), z(t))$ = constant for all values of t for which the solution $(x(t), y(t), z(t))$ of $\mathcal{X}$ is defined on U. Clearly H is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}H = 0$ on U. When H is a polynomial we say that H is a *polynomial first integral*.

For proving our main results concerning the existence of first integrals of Darboux type we shall use the invariant algebraic surfaces of system (2). This is the basis of the Darboux theory of integrability. The Darboux theory of integrability works for real or complex polynomial ordinary differential equations. The study of complex invariant algebraic curves is necessary for obtaining all the real first integrals of a real polynomial differential equation, for more details see [7–9,15,17].

Let $h = h(x, y, z) \in \mathbb{C}[x, y, z]$ be a non-constant polynomial. We say that $h = 0$ is an *invariant algebraic surface* of the vector field $\mathcal{X}$ in (3) if it satisfies $\mathcal{X}h = Kh$, for some polynomial $K = K(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of h. Note that K has degree at most 1. The polynomial h is called a *Darboux polynomial*, and we also say that K is the *cofactor* of the Darboux polynomial h. We note that a Darboux polynomial with zero cofactor is a polynomial first integral.

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Let $g, h \in \mathbb{C}[x, y, z]$ be coprime. We say that a non-constant function $E = e^{h/g}$ is an *exponential factor* of the vector field $\mathcal{X}$ given in (3) if it satisfies $\mathcal{X}E = LE$, for some polynomial $L = L(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of E and having degree at most 1. This relation is equivalent to

$$y\frac{\partial(g/h)}{\partial x} + z\frac{\partial(g/h)}{\partial y} + [-xz - \lambda(1-y^2)]\frac{\partial(g/h)}{\partial z} = K. \qquad (4)$$

For a geometrical and algebraic meaning of the exponential factors see [4].

A first integral G of system (2) is called of *Darboux type* if it is of the form

$$G = f_1^{\lambda_1}\ldots f_p^{\lambda_p}E_1^{\mu_1}\ldots E_q^{\mu_q},$$

where $f_1, \ldots, f_p$ are Darboux polynomials, $E_1, \ldots, E_q$ are exponential factors and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$, $k = 1, \ldots, q$. For more information on the Darboux theory of integrability see, for instance, [15,17] and the references quoted there.

The main result of this paper is the following:

Theorem 1. *For the Falkner–Skan and Blasius system the following statements hold.*

- (a) *Both systems have no polynomial first integrals.*
- (b) *The unique irreducible Darboux polynomial with nonzero cofactor of the Blasius system is z; the unique Darboux polynomial of the Falkner–Skan system is $1 - y^2 + 2xz$ if $\lambda = 1/2$.*
- (c) *The unique exponential factors of both systems are e^x and e^y, except if $\lambda = -1$ then we have the additional exponential factor e^{z+xy}.*
- (d) *Both systems have no first integrals of Darboux type.*

Theorem 1 is proved in the next section.

2. Proof of Theorem 1

We separate the proof of Theorem 1 in four propositions, one for every statement.

Proposition 2. *System (2) has no polynomial first integrals.*

Proof. Let h be a polynomial first integral of system (2). Then it satisfies

$$y\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} + [-xz - \lambda(1-y^2)]\frac{\partial h}{\partial z} = 0. \qquad (5)$$

Without loss of generality we can write $h = \sum_{j=1}^{n} h_j(x, y, z)$, where each $h_j = h_j(x, y, z)$ is a homogeneous polynomial of degree j, and $h_n \neq 0$.

Computing the terms of degree $n + 1$ in (5) we get

$$[-xz + \lambda y^2]\frac{\partial h_n}{\partial z} = 0.$$

Therefore $h_n = h_n(x, y)$.

Computing the terms of degree n in (5) we get that

$$y\frac{\partial h_n}{\partial x} + z\frac{\partial h_n}{\partial y} + [-xz + \lambda y^2]\frac{\partial h_{n-1}}{\partial z} = 0,$$

that is

$$h_{n-1} = g_{n-1}(x, y) + \frac{z}{x}\frac{\partial h_n}{\partial y} + \frac{y}{x^2}\log(xz - \lambda y^2)\left[\lambda y\frac{\partial h_n}{\partial y} + x\frac{\partial h_n}{\partial x}\right],$$

where $g_{n-1}(x, y)$ is a function in the variables x and y. Since h_{n-1} is a polynomial, we have

$$\lambda y\frac{\partial h_n}{\partial y} + x\frac{\partial h_n}{\partial x} = 0.$$

Therefore $h_n = h_n(x^{-\lambda}y)$. Since $h_n \neq 0$ is a homogeneous polynomial of degree $n \geqslant 1$, we have $\lambda = -p/q$ with p and q integers such that $p \geqslant 0, q \geqslant 1$, $p + q = n$ and $h_n = \alpha_n x^p y^q$, where $\alpha_n \in \mathbb{C}\setminus\{0\}$. Of course in the case of the Blasius system $p = 0$. In short we get that $h_n = \alpha_n x^p y^q$.

Now we have that

$$h_{n-1}(x, y, z) = \alpha_n q x^{p-1} y^{q-1} z + g_{n-1}(x, y).$$

So for the Blasius system ($p = 0$) we have a contradiction with the fact that $h_{n-1}(x, y, z)$ is a homogeneous polynomial of degree $n - 1$. Thus the proposition is proved for the Blasius system. In what follows we assume that $p \geqslant 1$, i.e. we restrict our attention to the Falkner–Skan system.

Computing the terms of degree $n - 1$ in (5) we get that

$$y\frac{\partial h_{n-1}}{\partial x} + z\frac{\partial h_{n-1}}{\partial y} + [-xz + \lambda y^2]\frac{\partial h_{n-2}}{\partial z} = 0,$$

that is

$$h_{n-2} = g_{n-2}(x, y) + \frac{1}{2}\alpha_n[2(p-q)y^2 + (q-1)qxz]x^{p-3}y^{q-2}z + \frac{z}{x}\frac{\partial g_{n-1}}{\partial y} + \frac{y}{qx^4}\log(py^2 + qxz)A,$$

where

$$A = -\alpha_n p(p-q)x^p y^{q+1} + qx^3\frac{\partial g_{n-1}}{\partial x} - px^2 y\frac{\partial g_{n-1}}{\partial y}.$$

Since h_{n-2} is a homogeneous polynomial of degree $n - 2$ we have that $A = 0$. Solving this linear partial differential equation we get that

$$g_{n-1}(x, y, z) = -\frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1} + G(x^{p/q}y).$$

Since $p + q = n$ and g_{n-1} is a homogeneous polynomial of degree $n - 1$ we have that

$$g_{n-1}(x, y, z) = -\frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1}.$$

Therefore

$$h_{n-1} = \alpha_n q x^{p-1}y^{q-1}z - \frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1},$$

$$h_{n-2} = g_{n-2}(x, y) + \frac{1}{2}\alpha_n[2(p-q)y^2 + (q-1)qxz]x^{p-3}y^{q-2}z - \frac{1}{p+2q}\alpha_n p(p-q)(q+1)x^{p-3}y^q z.$$

Note that h_{n-1} is a polynomial if $p \geqslant 1$ and $q \geqslant 1$, so $n \geqslant 2$; and that h_{n-2} is a polynomial if $p \geqslant 3$ and $q \geqslant 2$, so $n \geqslant 5$.

Working in a similar way with the terms of degree $n - 2$ in (5) we get that

$$g_{n-2} = \frac{\alpha_n x^{p-4}y^{q-2}}{2q(p+2q)^2}\Big[-q^2(p+2q)^2x^4 + p(p^3(q-1) - 6q^3 + pq^2(8+q) - p^2q(1+2q))y^4\Big]$$

and

$$h_{n-3} = \frac{1}{6(p+2q)^2}\Big[6(p+2q)^2 g_{n-3}(x, y) + \alpha_n x^{p-5}y^{q-3}z\Big(3(2-q)q(p+2q)^2x^4 + 3(p-4)(p-q)(6q^2 - p^2 - 2pq + p^2q - pq^2)y^4 + 3(-p+q)(p+2q)(p+2q-pq-4q^2+pq^2)xy^2z + (q-2)(q-1)q(p+2q)^2x^2z^2\Big)\Big]$$