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An intrusive hybrid method for discontinuous two-phase flow under uncertainty

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ABSTRACT

An intrusive stochastic projection method for two-phase time-dependent flow subject to uncertainty is presented. Numerical experiments are carried out assuming uncertainty in the location of the physical interface separating the two phases, but the framework generalizes to uncertainty with known distribution in other input data. Uncertainty is represented through a truncated multiwavelet expansion.

We assume that the discontinuous features of the solution are restricted to computational subdomains and use a high-order method for the smooth regions coupled weakly through interfaces with a robust shock capturing method for the non-smooth regions.

The discretization of the non-smooth region is based on a generalization of the HLL flux, and have many properties in common with its deterministic counterpart. It is simple and robust, and captures the statistics of the shock. The discretization of the smooth region is carried out with high-order finite-difference operators satisfying a summation-by-parts property.

1. Introduction

Physical models in computational fluid dynamics can be extended with stochastic models to represent uncertainty in governing equations or input parameters, including e.g. boundary and initial conditions, geometry and rates of reaction, diffusion and advection. One can distinguish between non-intrusive methods, where existing deterministic numerical solvers are called with a range of input values, and intrusive methods, resulting in a modified problem that is larger than the original deterministic problem, but only needs to be solved once. Despite the increased complexity of the reformulated problem, it has the potential to result in reduced computational cost compared to that of non-intrusive methods, such as Monte Carlo methods or stochastic numerical integration methods.

A stochastic two-phase problem in one spatial dimension is investigated as a first step towards developing an intrusive method for shock–bubble interaction with generic uncertainty in the input parameters. So et al. [1] investigated a two-dimensional two-phase problem subject to uncertainty in bubble deformation and contamination of the gas bubble, based on the experiments in [2]. The eccentricity of the elliptic bubble and the ratio of air–helium of the bubble were assumed to be random variables, and quantities of interest were obtained by numerical integration in the stochastic range (stochastic collocation). Previous work on uncertainty quantification for multi-phase problems include petroleum reservoir simulations with stochastic point collocation methods where deterministic flow solvers are evaluated at stochastic collocation points [3] and Karhunen–Loève expansions combined with perturbation methods [4].

We assume uncertainty in the location of the material interface, which requires a stochastic representation of all flow variables. Stochastic quantities are represented as generalized chaos series, that could be either global as in the case of generalized polynomial chaos (gPC) [5], or localized, see e.g. [6]. For robustness, we use a generalized chaos expansion with multiwavelets to represent the solution in the stochastic dimension [7]. It should be noted that this basis is global, so the method is fully intrusive. However, the basis is hierarchically localized in the sense that multiwavelets belonging to the same resolution level are grouped into families with non-overlapping support. These features makes it suitable for approximating discontinuities in the stochastic space without the emergence of spurious oscillations that occur in the case of global polynomial bases.

The stochastic Galerkin method is applied to the stochastic two-phase formulation, resulting in a finite-dimensional deterministic system that shares many properties with the original deterministic problem. The regularity properties of the stochastic problem are

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essential in the design of an appropriate numerical method. Chen et al. studied the steady-state inviscid Burgers' equation with a source term [8]. We used a similar approach for the inviscid Burgers' equation with uncertain boundary conditions and also analyzed the regularity of low-order stochastic Galerkin approximations of the problem [9]. Schwab and Tokareva analyzed regularity of scalar hyperbolic conservation laws and a linearized version of the Euler equations with uncertain initial profile [10]. In this paper, we analyze smoothness of the stochastic two-phase system.

The stochastic Galerkin problem is hyperbolic. This generalized and extended two-phase problem is solved with a hybrid method coupling the continuous phase region with the discontinuous phase region through a numerical interface. The non-smooth region is solved with the HLL-flux and MUSCL-reconstruction in space. The minmod flux limiter is employed in the numerical experiments displayed below.

Finite-difference operators in summation-by-parts (SBP) form are used for the high-order spatial discretization. A symmetrized problem formulation that generalizes the energy estimates in [11] for the Euler equations is used for the stochastic Galerkin system. The coupling between the different solution regions is performed with a weak imposition of the interface conditions through an interface using a penalty technique [12]. A fourth order Runge–Kutta method is used for the integration in time.

2. Representation of uncertainty

The polynomial chaos framework for uncertainty quantification introduced in [13] and generalized in [5] is used to represent uncertainty in the input parameters of the governing equations.

Let ω be an outcome of a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, with event space Ω, σ-algebra $\mathcal{A}$, and probability measure $\mathcal{P}$. Let $\boldsymbol{\xi} = \{\xi_j(\omega)\}_{j=1}^N$ be a set of N i.i.d. random variables for $\omega \in \Omega$. Each random variable ξ_j is a mapping from the event space to $\mathbb{R}$. For the cases presented here, $N = 1$ i.e. a single source of uncertainty is assumed but the framework can be generalized to multiple sources of uncertainty.

Consider a generalized chaos basis $\{\psi_i(\xi)\}_{i=0}^\infty$ spanning the space of second order (i.e. finite variance) random processes on this probability space. The basis functions are assumed to be orthonormal,

$$\langle\psi_i\psi_j\rangle = \int_\Omega \psi_i(\xi)\psi_j(\xi)d\mathcal{P}(\xi) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Second order random fields $u(x, t, \xi)$ can be expressed as

$$u(x,t,\xi) = \sum_{i=0}^\infty u_i(x,t)\psi_i(\xi). \tag{1}$$

Independent of the choice of orthogonal basis $\{\psi_i\}_{i=0}^\infty$, we can express mean and variance of $u(x, t, \xi)$ as

$$E(u(x,t,\xi)) = \int_\Omega u(x,t,\xi)d\mathcal{P}(\xi) = u_0(x,t),$$

and

$$\mathrm{Var}(u(x,t,\xi)) = \int_\Omega (u(x,t,\xi) - E(u(x,t,\xi)))^2 d\mathcal{P}(\xi) = \sum_{i=1}^\infty u_i^2(x,t),$$

respectively.

In the context of the stochastic Galerkin method, (1) is truncated to $M + 1$ terms, and we set

$$u(x,t,\xi) \approx \sum_{i=0}^M u_i(x,t)\psi_i(\xi).$$

The truncated generalized chaos solution converges in the $L_2(\Omega, \mathcal{P})$ norm to the exact solution as $M \to \infty$.

2.1. Multiwavelet expansion

Hyperbolic problems exhibit sharp gradients and shocks, for which polynomial representations are prone to fail, see e.g. [14,15]. For robustness, we follow the approach of [16] and use piecewise polynomial multiwavelets (MW), defined on sub-intervals of $[-1, 1]$. The construction of a truncated MW basis follows the algorithm in [17].

Wavelets are defined hierarchically on different resolution levels, representing successively finer features of the solution with increasing resolution. They have non-overlapping support within each resolution level, and in this sense they are localized. Still, the basis is global due to the overlapping support of wavelets belonging to different resolution levels. Piecewise constant wavelets, denoted Haar wavelets, do not exhibit spectral convergence, but avoid the Gibbs phenomenon in the proximity of discontinuities in the stochastic dimension.

Starting with the space $\mathbf{V}_{N_p}$ of polynomials of degree at most N_p defined on the interval $[-1, 1]$, the construction of multiwavelets aims at finding a basis of piecewise polynomials for the orthogonal complement of $\mathbf{V}_{N_p}$ in the space $\mathbf{V}_{N_p+1}$ of polynomials of degree at most $N_p + 1$. Merging the bases of $\mathbf{V}_{N_p}$ and that of the orthogonal complement of $\mathbf{V}_{N_p}$ in $\mathbf{V}_{N_p+1}$, we obtain a piecewise polynomial basis for $\mathbf{V}_{N_p+1}$. Continuing the process of finding orthogonal complements in spaces of increasing degree of piecewise polynomials, leads to a basis for $L_2([-1, 1])$.

We first introduce a smooth polynomial basis on $[-1, 1]$. Let $\{Le_i(\xi)\}_{i=0}^\infty$ be the normalized Legendre polynomials, defined recursively by

$$Le_{i+1}(\xi) = \sqrt{2i+3}\left(\frac{\sqrt{2i+1}}{i+1}\xi Le_i(\xi) - \frac{i}{(i+1)\sqrt{2i-1}}Le_{i-1}(\xi)\right), \quad i \geqslant 1,$$
$$Le_0(\xi) = 1, \quad Le_1(\xi) = \sqrt{3}\xi.$$

The set $\{Le_i(\xi)\}_{i=0}^{N_p}$ is an orthonormal basis for $\mathbf{V}_{N_p}$.

Following the algorithm by Alpert [17], we construct a set of *mother wavelets* $\{\psi_i^W(\xi)\}_{i=0}^{N_p}$ defined on the domain $\xi \in [-1, 1]$, where

$$\psi_i^W(\xi) = \begin{cases} p_i(\xi) & -1 \leqslant \xi < 0, \\ (-1)^{N_p+i+1}p_i(\xi) & 0 \leqslant \xi < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where $p_i(\xi)$ is an ith order polynomial. By construction, the set of wavelets $\{\psi_i^W(\xi)\}_{i=0}^{N_p}$ are orthogonal to all polynomials of order at most N_p, hence the wavelets are orthogonal to the set $\{Le_i(\xi)\}_{i=0}^{N_p}$ of Legendre polynomials of order at most N_p. Based on translations and dilations of (2), we get the wavelet family

$$\psi_{i,j,k}^W(\xi) = 2^{j/2}\psi_i^W(2^j\xi - k), \quad i = 0,\dots,N_p, \; j = 0,1,\dots,$$
$$k = 0,\dots,2^{j-1}.$$

Let $\psi_m(\xi)$ for $m = 0,\dots,N_p$ be the set of Legendre polynomials up to order N_p, and concatenate the indices i, j, k into $m = (N_p + 1)(2^j + k - 1) + i$ so that $\psi_m(\xi) \equiv \psi_{i,j,k}^W(\xi)$ for $m > N_p$. With the MW basis $\{\psi_m(\xi)\}_{m=0}^\infty$ we can represent any random variable $u(x, t, \xi)$ with finite variance as

$$u(x,t,\xi) = \sum_{m=0}^\infty u_m(x,t)\psi_m(\xi),$$

which is of the form (1). In the computations, we truncate the MW series both in terms of the piecewise polynomial order N_p and the *resolution level* N_r. With the index $j = 0,\dots,N_r$, we retain $M + 1 = (N_p + 1)2^{N_r}$ terms of the MW expansion (see Fig. 1).