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Parallel defect-correction algorithms based on finite element discretization for the Navier–Stokes equations

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ABSTRACT

Based on a fully overlapping domain decomposition technique and finite element discretization, two parallel defect-correction algorithms for the stationary Navier–Stokes equations with high Reynolds numbers are proposed and investigated. In these algorithms, each processor first solves an artificial viscosity stabilized Navier–Stokes equations by Newton or Picard iterative method, and then diffuses the system in the correction steps where only a linear problem needs to be solved at each step. All the computations are performed in parallel on global composite meshes that are fine around a particular subdomain and coarse elsewhere. The algorithms have low communication complexity. They can yield an approximate solution with an accuracy comparable to that of the standard finite element solution. Numerical tests demonstrated the effectiveness of the algorithms.

1. Introduction

Large scale computational fluid dynamics problems need large computational resources that probably can only be provided by parallel computers or a cluster of workstations. Therefore, with the popularity of parallel and multicore computers, parallel computations attract more and more attentions in computational fluid dynamics community and much effort has been throwing into the development of efficient parallel computing methods for the Navier–Stokes equations and related flow problems (cf. [1–8]).

For high Reynolds number flows, due to the domination of the convective term in the Navier–Stokes equations, the standard discretization schemes such as the standard finite element and finite difference methods may cause spurious oscillations. What is worse, the iterative method used to solve the nonlinear system may fail to converge on a mesh that is not fine enough, and hence cannot yield an approximate solution (cf. [9–13]). Therefore, to simulate high Reynolds number flows, stabilization techniques such as the subgrid-scale model methods (cf. [14–17]), the variational multiscale methods (cf. [18–21]) and the defect-correction method (cf. [9,10]) should be adopted. Among these stabilization methods, the defect-correction method attracted many attentions due to its good efficiency and simplicity of implementation. It is an iterative improvement technique for increasing the accuracy of a computed solution without refinement of the underlying mesh. Tayton [9,22] first employed it to solve the Navier–Stokes equations. It was then incoperated with the adaptive technique [23], the two-level method [24] and the characteristic finite element method [26]. It was also applied to the viscoelastic fluid flow problems [25] and the conduction convection problems [27–29]. We refer to [15] and [27] for a short literature review of the defect-correction method. For the Navier–Stokes equations, the defect-correction method first solves the nonlinear Navier–Stokes equations with an added artificial viscosity term on a relatively coarse grid, and then corrects the solution on the same grid where only a linear problem is involved at each correction step. It incorporates the artificial viscosity term as a stabilizing factor, making the nonlinear system easier to solve based on existing codes. The reader is referred to [9,10] for the details.

In this paper, a combination method of the defect-correction approach with a parallelization technique based on fully overlapping domain decomposition [30] is applied to numerically solve the Navier–Stokes equations. This parallelization technique is, in a way, related to the *full domain partition* presented by Mitchell in [31,32] as an approach to distributing adaptive grids. Similar approaches were also proposed in literature. For example, based on the understanding of the local and global properties of a finite element solution to some elliptic problems, Xu and Zhou [33] proposed a similar parallel approach to discretizing a class of linear and nonlinear elliptic boundary value problems, and gave detailed error analysis. In 2001, Bank and Jimack [34] presented a parallel preconditioner for discretized system of elliptic partial differential equations based on a similar approach. In [30], we discussed a

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parallel finite element discretization method based on fully overlapping domain decomposition technique for the stationary incompressible Navier–Stokes equations. In this method, each processor computes a local solution to the Navier–Stokes equations in its own subdomain using a global composite mesh that is fine around its own subdomain and coarse far away from the subdomain. Theoretical analysis and numerical tests showed that, with a suitable ratio of coarse mesh size H to fine mesh size h, this method can yield an approximate solution with an accuracy comparable to that of the standard finite element solution. It has low communication complexity and can be implemented easily based on an existing sequential solver. However, due to the coarseness of the grid far away from the interested subdomain, it is challenging for the method to simulate high Reynolds numbers flows as the standard finite element methods encountered. To circumvent this difficulty, we combine this method with the defect-correction method and then design two parallel defect-correction finite element algorithms for the stationary Navier–Stokes equations which are particularly efficient and combine the best algorithmic features of each method. Concretely, we decompose the solution domain into subdomains with each processor assigned one subdomain. Each processor first independently generates a global composite grid that is fine with size h around its own subdomain and coarse with size $H \gg h$ far away from the subdomain, and then computes a local solution in its own subdomain using defect-correction method on this global composite grid. Each subproblem in these algorithms is actually a global problem defined in the entire domain with the majority of the degrees of freedom associated with the particular subdomain that it is responsible for, and hence can be solved in parallel with other subproblems using an existing sequential defect-correction solver without extensive recoding.

The rest of the paper is organized as follows. In the next section, the Navier–Stokes equations and their mixed finite element approximations are provided. In Section 3, the defect-correction method and the fully overlapping domain decomposition technique are first briefly described. Combining the best algorithmic features of these two methods, two parallel defect-correction algorithms are then designed and investigated. Numerical tests are given in Section 4 to demonstrate the effectiveness of the algorithms. Finally, conclusions are drawn in Sections 5.

2. Preliminaries

Let Ω be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ in $\mathbb{R}^d (d = 2, 3)$. As usual, for a nonnegative integer k, we denote by $H^k(\Omega)$ the Sobolev space of functions with square integrable distribution derivatives up to order k in Ω, equipped with the standard norm $\|\cdot\|_{k,\Omega}$, while denote by $H_0^1(\Omega)$ the closed subspace of $H^1(\Omega)$ consisting of functions with zero trace on $\partial\Omega$; see, e.g., [35,36].

2.1. The Navier–Stokes equations

We consider the following incompressible Navier–Stokes equations

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \mathrm{div}\, u = 0 \quad \text{in } \Omega, \tag{2.1}$$

where $u = (u_1, \ldots, u_d)^T$ is the velocity, p the pressure, $f = (f_1, \ldots, f_d)^T$ the prescribed body force and ν the kinematic viscosity. Given a characteristic length L and a characteristic velocity U, the Reynolds number is defined as $Re = UL/\nu$.

The above Eqs. (2.1) need to be supplemented by some boundary conditions according to the physical situations; for example, the Dirichlet boundary condition

$$u = g \quad \text{on } \partial\Omega, \tag{2.2}$$

with $g : \Omega \to \mathbb{R}^d$ satisfying $\int_{\partial\Omega} g\cdot n dx = 0$ (here n denotes the unit outward normal vector to $\partial\Omega$), or the mixed boundary condition

$$u = g_0 \quad \text{on } \Gamma_0, \quad \nu\frac{\partial u}{\partial n} - np = g_1 \quad \text{on } \Gamma_1, \tag{2.3}$$

where Γ_0 and Γ_1 are two subsets of $\partial\Omega$ satisfying $\Gamma_0 \cap \Gamma_1 = \phi$ and $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, g_0 and g_1 are two given functions. For more complicated boundary conditions, the reader is referred to, for example, [37,38]. For the simplicity of presentation, we just consider *no-slip* boundary condition (namely, the boundary condition (2.2) with $g = 0$), while other boundary conditions will be considered in the numerical tests.

To introduce the variational formulation of (2.1) supplemented with the *no-slip* boundary condition, we set

$$X = H_0^1(\Omega)^d, \quad Y = L^2(\Omega)^d, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\},$$

and define bilinear terms $a(\cdot,\cdot), \tilde{d}(\cdot,\cdot)$ and trilinear term $b(\cdot,\cdot,\cdot)$ as

$$a(u, v) = (\nabla u, \nabla v), \quad \tilde{d}(v, q) = (\mathrm{div}\, v, q), \quad \forall\, u, v \in X, \quad q \in M,$$

$$\begin{aligned} b(u, v, w) &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\mathrm{div} u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v), \quad \forall\, u, v, w \in X, \quad q \in M, \end{aligned}$$

where $(\cdot,\cdot)$ is the standard inner-product of $L^2(\Omega)^l$ ($l = 1, 2, 3$).

With the above notations, the variational formulation of (2.1) reads: find a pair $(u,p) \in X \times M$ such that

$$\nu a(u, v) + b(u, u, v) - \tilde{d}(v, p) + \tilde{d}(u, q) = (f, v), \quad \forall (v, q) \in X \times M. \tag{2.4}$$

Defining

$$N = \sup_{\substack{u, v, w \in X, \\ u, v, w \neq 0}} \frac{|b(u, v, w)|}{\|\nabla u\|_{0,\Omega}\|\nabla v\|_{0,\Omega}\|\nabla w\|_{0,\Omega}},$$

we have the following existence and uniqueness results (cf. [39,40]).

Lemma 2.1. *Given $f \in X'$ (the dual space of X), there exists at least a solution pair $(u,p) \in X \times M$ satisfying (2.4) and*

$$\|\nabla u\|_{0,\Omega} \leqslant \nu^{-1}\|f\|_{-1,\Omega}, \quad \|f\|_{-1,\Omega} = \sup_{\substack{v \in X, \\ v \neq 0}} \frac{|(f, v)|}{\|\nabla v\|_{0,\Omega}}. \tag{2.5}$$

Moreover, if ν and f satisfy the following uniqueness condition

$$\frac{N\|f\|_{-1,\Omega}}{\nu^2} < 1, \tag{2.6}$$

then the solution pair (u,p) of problem (2.4) is unique.

2.2. Mixed finite element approximation

To describe the mixed finite element approximations of problem (2.4), let us assume $T^h(\Omega) = \{K\}$ to be a shape-regular triangulation (see, e.g., [36,40]) of Ω into triangles or quadrilaterals (if $d = 2$), or tetrahedrons or hexahedrons (if $d = 3$) with mesh-size function $h(x)$ whose value is the diameter h_K of the element K containing x, satisfying the following assumption:

A0. Triangulation. There exists a constant $\gamma \geqslant 1$ such that

$$h_\Omega^\gamma \leqslant ch(x), \quad \forall x \in \Omega, \tag{2.7}$$

where $h_\Omega = \max_{x\in\Omega} h(x)$ is the largest mesh size of $T^h(\Omega)$.