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Analytical solution for strain gradient elastic Kirchhoff rectangular plates under transverse static loading

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ABSTRACT

Levy's analytical solution approach is extended for analysis of rectangular strain gradient elastic plates under static loading for the first time with different boundary conditions at the edges using the method of superposition. The governing equation of equilibrium and the corresponding classical/non-classical boundary conditions for strain gradient flexural Kirchhoff plate under static loading are considered. Numerical examples on static bending of Kirchhoff nanoplates involving five different combinations of simply supported, clamped and free edge boundary conditions are presented. The effect of negative strain gradient terms is of hardening nature thus resulting in decrease in the deflection. Plates with geometry comparable to the microstructural length scale show significant size effect and this size dependency diminishes with the increase in the plate size.

1. Introduction

Micro and nano sized structural components, such as plates and shells find extensive applications in Microelectromechanical systems (MEMS) and Nanoelectromechanical systems (NEMS) as vibration/mass/gas/force sensors, atomistic dust detectors and actuators/resonators. The support conditions in these applications can be SSSS/SCSC/SFSF/SCSF/CCCC (Bunch et al., 2007; Sakhaee-Pour et al., 2008; Arash et al., 2011; Shen et al., 2012). However, their mechanical behavior is affected by the material microstructure and long-range interaction forces which are comparatively weaker at macroscales. Some of the recent experiments (Stölken and Evans, 1998; Lam et al., 2003; McFarland and Colton, 2005) on various materials at micro and nanoscales have captured these size effects to a significant extent. The forces between atoms depend not only on the stretching, bending and twisting of the bonds but also on the long-range interactions between non-adjacent atoms such as van der Waals interaction. The classical continuum theories are unable to capture these size effects at micro/nano scale as they do not contain any internal length scale parameter. Atomistic/molecular modelling (Allinger, 1977; Lii and Allinger, 1989; Tersoff, 1988; Brenner, 1990) can capture such nonlocal interactions associated with atomic or molecular motions, however it is computationally prohibitive for fairly large size structures.

Several modifications in the classical elasticity formulation have been proposed to address the small scale effects and are classified as: stress gradient, strain gradient, couple stress and integral types. Their predictions reduce to those of local continuum theories when the domain is much larger than the internal length scale. Eringen's stress gradient nonlocal theory (Eringen, 1972; Eringen and Edelen, 1972) assumes that the stress at a point is considered as a function of the strain fields at all the points in the entire domain which is governed by a distance decaying attenuation function. The atomic length scales are directly introduced into the constitutive equations as material parameters (Eringen, 1983). This theory is the most popular among all nonlocal theories and finds numerous applications in the recent literature (Zhang et al., 2014; Yu et al., 2016; Karličić et al., 2017; Yan et al., 2017; Nazemnezhad et al., 2018). Different higher-order strain gradient and inertia gradient theories have been proposed (Aifantis, 1992; Ru and Aifantis, 1993; Chang and Gao, 1995; Mühlhaus and Oka, 1996) to capture the size effect prominent at the micro/nanoscale. Combining the Eringen's stress gradient model and the second-order strain gradient model with negative coefficient, a hybrid gradient elasticity model (Gutkin and Aifantis, 1999; Aifantis, 2003) was proposed to avoid singularities in both stress as well as strain fields. Peridynamic elasticity theory (Silling, 2000; Silling et al., 2003) leads to integral form of the governing equations of motion rather than the partial differential equations. However, the absence of contact force between adjacent elements corresponds to discontinuous displacement field when concentrated forces are applied at the boundary.

By introducing an additional higher order equilibrium equation involving moment of couples along with the traditional equilibrium equations, the classical couple stress theory (Mindlin and Tiersten,

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1962; Toupin, 1962; Mindlin, 1964) was reduced to the modified couple stress theory (Yang et al., 2002). Modified strain gradient theory (Lam et al., 2003) was another development in the field of nonlocal elasticity which considered the strain energy density as a function of the symmetric strain, dilatation gradient, deviatoric stretch gradient and symmetric rotation gradient tensors. The mechanically based approach to nonlocal elasticity theory (Di Paola et al., 2009, 2013a) modelled interactions between adjacent volume elements as classical contact forces while long-range interactions between non-adjacent elements were modelled as distance-decaying central body forces proportional to the relative displacements and regulated by a suitably chosen attenuation function. This theory has been employed for investigation of beams (Di Paola et al., 2011, 2013b), wave propagation in rods (Zingales, 2011) and plates supported on subgrade (Failla et al., 2013). Among all these theories, the second-order strain gradient theories are the easiest to formulate, include only one nonlocal parameter unlike the modified strain gradient model (Lam et al., 2003) and are capable of representing the nonlocal effect for different loading/boundary conditions unlike the Eringen's theory (Reddy and Pang, 2008). Strain gradient model with positive nonlocal coefficient (Kumar et al., 2008; Ansari et al., 2012; Hosseini-Ara et al., 2012) does not satisfy uniqueness and stability of the solution for certain range of the nonlocal parameter (Askes and Aifantis, 2006, 2011; Papargyri-Beskou et al., 2009), whereas its negative counterpart is unconditionally stable and is the topic of interest for this study.

To smoothen the discontinuities and singularities in the stress and/or strain field near imperfections, a second-order negative strain gradient theory was proposed (Aifantis, 1992, 1994; 1999; Altan and Aifantis, 1992, 1997; Ru and Aifantis, 1993; Askes et al., 2002; Askes and Aifantis, 2006). The constitutive relations for negative second-order strain gradient nonlocal theory can be expressed as:

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - l^2\varepsilon_{kl,mm}) \tag{1}$$

where σ_{ij}, ε_{kl} and C_{ijkl} are the stress, strain and stiffness terms and l is the nonlocal parameter. The constitutive relation for the negative strain gradient theory can also be derived employing the second-order homogenization scheme (Gitman et al., 2005). Static and dynamic analyses of gradient elastic bar were carried out in tension (Tsepoura et al., 2002) using this theory. Static, stability and dynamic analyses of simply supported and cantilever isotropic Euler-Bernoulli beam (Papargyri-Beskou et al., 2003) and simply supported isotropic Kirchhoff plate (Papargyri-Beskou and Beskos, 2008) were performed analytically using second-order negative strain gradient theory. Static, stability and dynamic analyses of both ends simply supported, clamped-clamped, clamped-simply supported and cantilever CNT modelled as isotropic Euler-Bernoulli beam (Babu and Patel, 2018) were carried out through both analytical method and FEM using second-order positive/negative strain gradient theories. Softening and hardening effects were observed for positive and negative strain gradient theories, respectively.

To the best of the authors' knowledge, the work on strain gradient elastic plates is meager with limited studies on all edges simply supported plates. The analytical solution for nonlocal plates with other boundary conditions incorporating the strain gradient effect is not available in the literature. The need for analytical solutions in handling problems involving strain gradient elastic plates propels the present study.

The paper is organized as follows: In Section 2, the governing equation of equilibrium and corresponding boundary conditions of rectangular isotropic Kirchhoff plate with second-order negative strain gradient effect are given. The solutions are derived analytically for boundary value problems involving five different boundary conditions by Levy's approach. In Section 3, numerical examples are presented to show the capability of the proposed solution for static bending of flexural plates under uniformly distributed loading condition and the gradient effect on the static response of the plates are assessed. Section 4 gives an overall summary of the present work.

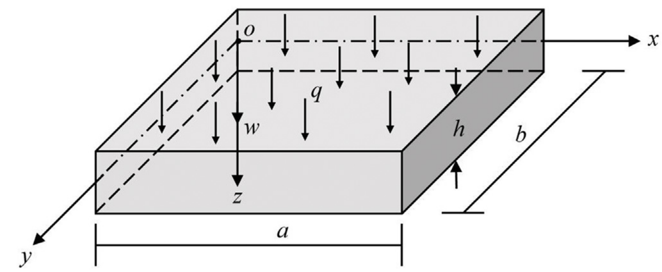

Fig. 1. Plate geometry and coordinate system.

2. Formulation

A thin rectangular isotropic plate subjected to lateral uniformly distributed load q with length along x-axis as a, width along y-axis as b and uniform thickness h is considered (Fig. 1). Based on the Kirchhoff's flexural plate theory (Timoshenko and Woinowsky-Krieger, 1959), displacements u, v and w along x, y and z directions, respectively are written as:

$$\begin{aligned} u(x, y, z) &= -z\frac{\partial w(x, y)}{\partial x} \\ v(x, y, z) &= -z\frac{\partial w(x, y)}{\partial y} \\ w(x, y, z) &= w(x, y) \end{aligned} \tag{2}$$

where w is the transverse displacements of the point $(x, y, 0)$ on the mid-plane of the plate. The strain-displacement relations can be written as (Timoshenko and Woinowsky-Krieger, 1959):

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2},\ \ \varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2},\ \ \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{3}$$

where ε_{xx} and ε_{yy} are the normal strains in x and y directions, respectively and γ_{xy} is the in-plane shear strain. The equation of equilibrium is derived as (Timoshenko and Woinowsky-Krieger, 1959):

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q = 0 \tag{4}$$

where M_{xx} and M_{yy} are the bending moment resultants about positive y and negative x directions, respectively and M_{xy} is the twisting moment resultant about negative x direction. The sign convention followed for moments while deriving Eq. (4) is as per the sense of moments about the mid-plane axes produced by positive stresses over a differential area on positive z side. The moment resultants are defined as (Timoshenko and Woinowsky-Krieger, 1959):

$$\begin{aligned} M_{xx} &= \int_{-h/2}^{h/2} \sigma_{xx} z dz \\ M_{yy} &= \int_{-h/2}^{h/2} \sigma_{yy} z dz \\ M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z dz \end{aligned} \tag{5}$$

where σ_{xx} and σ_{yy} are the normal stresses in x and y directions, respectively and τ_{xy} is the in-plane shear stress.

2.1. Second-order strain gradient nonlocal theory

Under plane stress condition, Eq. (1) can be written in terms of Cartesian coordinates x and y as (Papargyri-Beskou and Beskos, 2008; Papargyri-Beskou et al., 2010):