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Exact solution for the mode III stress fields ahead of cracks initiated at sharp notch tips

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A R T I C L E I N F O

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A B S T R A C T

In this work, the exact solution for the stress fields ahead of cracks initiated at sharp notch tips under antiplane shear and torsion loadings is derived in close form, leveraging conformal mapping and the complex potential method for antiplane elasticity.

Based on the stress field distributions, relevant expressions for the mode III crack stress intensity factors are derived and their accuracy is discussed in detail taking advantage of a bulk of results from FE analyses.

1. Introduction

Stress concentrators, such as notches and holes, are unavoidably present in mechanical components, leading to cracking phenomena both under static and fatigue loadings.

Explicitly or implicitly, local approaches to the static and fatigue design of mechanical components are based on the local stress fields close to notch tips (see among the others, Yosibash and Mittelman, 2016, and references reported therein) thus justifying the large attention paid over the decades in the literature to the study of this problem (see, just to mention a few: Inglis, 1913, Neuber, 1958, Creager and Paris, 1967, Kullmer, 1992, Lazzarin and Tovo, 1996, Zappalorto et al., 2010, Zappalorto and Lazzarin, 2011, Feriani et al., 2011, Felger and Becker, 2017, Zappalorto and Maragoni, 2018).

When dealing with cyclic loadings, notch tip stresses are thought of as controlling the fatigue life spent to initiate short cracks fully immersed in the stress field of the un-cracked (notched) component. For some relevant applications, such as the design against fatigue of steel and aluminum welded joints, the initiation life is the major part of the entire life of the component and fatigue life predictions can be simply based on Williams' asymptotic stress field distribution around the uncracked weld toe or root region (Livieri and Lazzarin, 2005).

In the case of notched (unwelded) components, instead, the ratio between the number of cycles to crack initiation and those to failure strictly depends on the notch tip radius. For large radii, the initiation phase is predominant, whereas, when the notch tip radius is very small, the propagation phase becomes more and more important (Lazzarin et al., 1997). Similar arguments can be used also in the case of manufacturing defects (see, among the others, Atzori et., 2003, Carraro et al., 2015, Maragoni et al., 2016). In all these cases, the propagation phase cannot be neglected, and should be assessed taking advantage of the integration of the Paris' fatigue curve. To this end, the knowledge of the crack stress intensity factor is essential and many authors devoted great efforts to determine K values for cracks emanating from notches.

The literature on this topic is so broad that a comprehensive review is far from easy and, especially, is out of the specific aim of the present paper. In the following, only some examples will be discussed, without the ambition to be thorough. Worth of being mentioned is the paper by Bowie (1956), who gave the solutions for a circular hole with a single edge crack and a pair of symmetrical edge cracks in a plate under tension, whilst Tweed and Rooke (1973) used the Mellin transform technique to study the case of a branching crack emanating from a circular hole under biaxial tension. The analysis was also extended to elliptical holes by Newman (1971) and Murakami (1978) who studied the tension problem for an elliptical hole with symmetrical edge cracks, and Isida and Nakamura (1980) who analysed a slant crack emanating from an elliptical hole under far applied uniaxial tension and shear.

The problem of cracks initiating at edge rounded notches was also comprehensively debated, amongst the others, by Lukas and Klesnil (1978), Bandyopadhyay and Deysarker (1981), Shijve (1982), Kujawskii (1991) and Xu et al. (1997) whereas cracks from sharp notches or holes were studied by Neal (1970), Muki and Westman (1974), Hasebe and Ishida (1978) and Hasebe and Ueda (1980), just to mention a few.

Despite such an intense scrutiny carried out in the previous century, this topic is still attracting the interests of researchers, as documented

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by some works published in the more recent literature related to cracks emanating from rounded notches (see for example Jones and Peng, 2002, Xiangqiao Yan, 2004, 2005; 2006; The et al., 2006; Abdelmoula et al., 2007; Weiβgraeber et al., 2016a,b) as well as from singular points (Philipps et al., 2008; Iida and Hasebe, 2016; Weiβgraeber et al., 2016a,b).

However, in the best of the authors' knowledge, the analytical study of the whole stress fields ahead of cracks nucleated at the tip of notches, accounting for the combined effect of the crack tip singularity and notch stresses, has received far less attention so far. Within this context, worth of mention is the work by Hasebe and Ishida (1978) who provided the solution for a crack originating from a triangular notch on a rim of a semi-infinite plate, and the paper by Hasebe and Ueda (1980) who studied the stress field of a crack originating from a square hole corner. In both cases, the proposed solution is exact but very complicated, and a simple expression for the stress field was not given in an explicit form.

The aim of the present paper is to partially fill this gap providing an exact yet simple stress field solution for a crack emanating from a pointed V notch in a plate subjected to antiplane shear loading and in a solid bar under torsion.

To this end, the conformal mapping technique is used in combination with a recent approach proposed by the authors (Salviato and Zappalorto, 2016), according to which the exact mode III stress field can be determined using the complex potential approach applied to the first derivative of the conformal mapping function.

Two relevant cases are addressed and solved separately:

- the mode III problem of a finite crack nucleating from a deep (mathematically infinite) pointed V notch in a body with a finite ligament;
- the mode III problem of a finite crack nucleating from a finite depth pointed V notch in a body with an infinite ligament.

Relevant, exact, expressions for the mode III crack stress intensity factors are also provided.

The accuracy of the proposed solutions is discussed, taking advantage of the comparison with the results from FE analyses carried out on elastic bodies subjected to antiplane shear and torsion loadings, showing a very satisfactory agreement also in the case of fully finite bodies.

2. Preliminary remarks

Consider a body made of a homogenous and isotropic material obeying the theory of linear elastic deformations. Further, consider the Cartesian reference system (x,y,z) represented in Fig. 1 and suppose that the body is loaded by a remote shear stress τ_∞ resulting only in displacements w in the z direction, normal to the plane of the notch characterized by the x and y axes (Fig. 1a).

Let us consider a notch profile and a conformal map $z = z(\xi)$ with

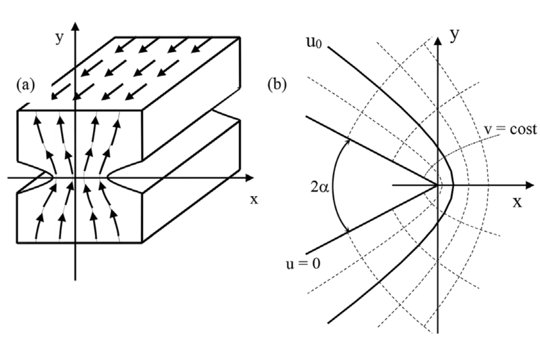

Fig. 1. (a) Notched body under longitudinal shear; (b) Typical conformal mapping describing the notch. The boundary is defined by the condition $u = u_0$.

$\xi = u + iv$ and $z = x + iy$ such that the notch profile is described by the condition $u(x, y) = u_0$ (Fig. 1b). The constant u_0 is taken as a positive number, so that the domain of integration belongs to the right half plane of the (u,v) space.

In the foregoing conditions, the out-of-plane displacement component w is harmonic (Timoshenko and Goodier, 1970) whereas the other components are equal to zero. Thanks to the properties of conformal mapping, the harmonicity of the displacement function $\omega(u, v) = w\{x(u, v), y(u, v)\}$ with respect to the curvilinear coordinates u and v is left intact (Greenberg, 2001; Salviato and Zappalorto, 2016):

$$\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial v^2} = 0 \tag{1}$$

The foregoing equation can be solved assuming separation of variables in curvilinear coordinates:

$$\omega = f(u)g(v) \tag{2}$$

Substituting Eq. (2) into (1) leads to:

$$f''(u)g(v) + f(u)g''(v) = 0 \tag{3}$$

or, equivalently:

$$\frac{f''(u)}{f(u)} = \frac{g''(v)}{g(v)} = \lambda^2 \tag{4}$$

where λ is real a constant. Accordingly, the governing PDE can be simplified into two ODE in the variables u, v:

$$f''(u) - \lambda^2 f(u) = 0 \quad g''(v) + \lambda^2 g(v) = 0 \tag{5}$$

With the aim to introduce the relevant boundary conditions, the following expressions for strains and stresses in curvilinear coordinate are useful (Sokolnikoff, 1983):

$$\gamma_{iz} = \frac{1}{h_i}\frac{\partial\omega}{\partial\alpha_i} \quad \tau_{iz} = \frac{G}{h_i}\frac{\partial\omega}{\partial\alpha_i} \tag{6}$$

where G is the elastic modulus in shear and h_i is the factor of distortion (Neuber, 1958).

As, in general, $h_i \neq 0$, the *Dirichlet conditions* in terms of stresses result in *Von Neuman* conditions on ω.

The problem is then defined by the following system of equations:

$$\begin{cases} g''(v) + \lambda^2 g(v) = 0 \\ f''(u) - \lambda^2 f(u) = 0 \\ f'(u_0) = 0 \\ \frac{|f'(u)|}{h_u} < \infty \quad for \quad u, v \to \infty \\ \frac{|g'(v)|}{h_v} < \infty \quad for \quad u, v \to \infty \end{cases} \tag{7a–e}$$

where Eq. (7c) is the free-of-stress condition along the notch edge, whereas Eq. (7d,e) express the condition for bounded stresses far away the notch tip.

The case $\lambda^2 \neq 0$ can be disregarded as it provides trivial solutions only. Differently, under the condition $\lambda^2 = 0$, the general solution:

$$\omega(u, v) = (A + Bu)(C + Dv) \tag{8}$$

can be further simplified into Eq. (9) to account for boundary conditions:

$$\omega(u, v) = C_1 + C_2 v \tag{9}$$

where C_1 represents a rigid translation which does not contribute to the strain field and can be ignored.

Introducing Eq. (9) into the definition of stresses and invoking Cauchy-Riemann conditions, one gets the following expressions:

$$\tau_{zx} = \varsigma\frac{\partial v}{\partial x} \quad \tau_{zy} = \varsigma\frac{\partial u}{\partial x} \tag{10}$$

$\varsigma = GC_2$ being a constant to determine. It is worth noting that, as $\xi' =$ d