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# A modified higher-order theory for FG beams

Y.L. Pei, P.S. Geng, L.X. Li*

*State Key Laboratory for Strength and Vibration of Mechanical Structures, Shaanxi Key Laboratory of Environment and Control for Flight Vehicle, School of Aerospace, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, China*

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ABSTRACT

Functionally graded (FG) beams are widely used in many fields. However, the corresponding beam theory is not well established. This paper begins with distinguishing the centroid and the neutral point of cross section. First, the deformation mode is mathematically suggested for axial displacement as a general higher-order form, and then orthogonally decomposed with the help of shear stress free conditions and definitions of generalized displacements (i.e. deflection, rotation and stretch). On this basis, the generalized stresses are defined together with the work conjugated generalized strains, and the decoupled constitutive relations are then derived. Next, the principle of virtual work is proposed for beam problems, and the variationally consistent higher-order theory is established for FG beams, which is as simple as that for a homogeneous beam. Finally, the present theory is demonstrated by typical FG beam problems for both the simply supported case and the clamped case. It is indicated that the analytical solution to the present modified higher-order theory can be regarded as the benchmark of FG beam problems. Furthermore, the relation with the traditional higher-order theory is clarified, which is beneficial to conduct a comparative study on different higher-order beam theories.

## 1. Introduction

Functionally graded (FG) materials have many advantages. For example, they hold good thermo-mechanical properties (Wetherhold et al., 1996) and can resist initiation and propagation of a crack (Adámek and Valeš, 2015). FG beams are widely used in many fields such as mechanical engineering, biomechanical engineering, automotive and aerospace industries. By FG beam we mean here the isotropic but non-homogenous beam (Reddy, 2011) with the elastic constants smoothly varying over the cross section.

FG beams have been extensively studied by using the existing beam theories. Using the Euler-Bernoulli theory (EBT), Sankar (2001) solved static response of FG beam subjected to transverse loads, and Pradhan and Chakraverty (2013) investigated free vibration via the Rayleigh-Ritz method for a material varying along the thickness of beam as a power law. By using the same hypotheses as the EBT, namely the Kirchhoff plate theory, Ghayesh et al. (2018) studied nonlinear oscillations of FG microplates considering the size effect.

To take into account the shear effect, the Timoshenko beam theory (TBT) in (Timoshenko, 1921) was applied to FG beams as well. With the TBT, Adámek and Valeš (2015) developed an analytical solution for a simply supported FG beam subjected to a transverse load for a material varying as an even function through the thickness of beam. Recently, considering the size effect, the TBT was used to study vibration and post-buckling of FG micro-beams (Li et al., 2016; Wu et al., 2017; Chen et al., 2017). The TBT is however controversial for determining the shear correction factor of FG beams (Adámek and Valeš, 2015; Frikha et al., 2016). To this end, higher-order beam theories were pursued instead.

In 2000, Reddy (2000) developed a higher-order theory for FG plates to study thermo-mechanical coupling and the Von Karman geometric nonlinearity under a transversely distributed load, and formulated the finite element method by using the third-order shear deformation plate theory in Reddy (1984). Recently, using the same higher-order theory, Oscillations of FG microbeams were studied (Ghayesh et al., 2017) by taking into account the size effect. Using a unified shear deformation theory (USDT) developed by Soldatos and Timarci (1993) in which the shape function covers many modes such as third-order mode in Reddy (1984), Ambartsumian (1958), Reissner (1975), sine mode in Touratier (1991), hyperbolic sine mode in Soldatos (1992) and exponential mode in Karama et al. (2003), Aydogdu and Taskin (2007) investigated free vibration of simply supported FG beam with Young's modulus varying in the thickness direction respectively as a power law and as an exponential law. Also based on the USDT, bending behaviors of hybrid FG beams and sigmoid FG beams were respectively investigated in Benatta et al. (2009), Ben-Oumrane et al. (2009). Also based on the same axial displacement mode, post-bucking of FG nanobeams (Khorshidi et al., 2016), size

* Corresponding author.
*E-mail address:* luxianli@mail.xjtu.edu.cn (L.X. Li).

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dependency in post-buckling of FG nanoshells (Sahmani and Aghdam, 2017), bending, buckling and vibration of temperature-dependent FG rectangular plates (Dong and Li, 2016) and size dependency of FG nanoplates (Phung-Van et al., 2017) were investigated, respectively. With another unified approach, Thai and Vo (2012) developed various higher-order shear deformation beam theories for bending and free vibration of FG beams while Ebrahimi and Barati (2017a, 2017b) investigated buckling of curved FG nanobeams and vibration of viscoelastic FG nanobeams.

It is clear that the existing work on FG beams is the direct extension of the homogeneous beams/plates. For example, axial displacement and moment for FG beams/plates (e.g. in Reddy, 2000; Aydogdu and Taskin, 2007; Thai and Vo, 2012) are identical to those for a uniform beam (e.g. in Reddy, 1984; Vo and Thai, 2012; Aydogdu, 2009). As a consequence, the coupling between bending and stretch which does not exist for the uniform beam appears unexpectedly for FG beams. For example, Eqs. (1) and (17) in Reddy (2000), Eqs. (5) and (9) in Aydogdu and Taskin (2007), and Eqs. (1) and (5) in Thai and Vo (2012) are all coupled for bending and stretch.

The fact is however that this coupling is not intrinsically present for FG beams (Morimoto et al., 2006; Abrate, 2008), and, Zhang and Zhou (2008) have successfully removed the stretch-bending coupling from the classic theory of FG plates by employing the physical neutral surface, though the coupling between the two bending terms was still present (Zhang, 2013a; She et al., 2017).

As a matter of fact, even for a uniform beam, use of inappropriate rotation (e.g. the one at neutral point) may cause coupling between bending and higher-order bending (e.g. in Levinson, 1980 and Reddy, 1984). In addition, translation of coordinate system can cause non-physical coupling between bending and stretch (e.g. see Byskov, 2013), which violates the frame-indifferent objectivity of physical quantities.

Motivated by the great importance to overcome aforementioned disadvantages, this paper begins with the fundamentals of beam problems to develop a modified higher-order theory for homogeneous beams, as well as for FG beams. To this end, this paper is organized as follows. In Section 2, a beam problem is described by setting the coordinate system and defining the generalized displacements of a FG beam. The deformation mode is suggested for a FG beam by analyzing the assumptions and conditions in Section 3. In Section 4, the generalized stresses are defined based on the deformation mode in Section 3, and the constitutive relations are further derived for a FG beam. In Section 5, the principle of virtual work is proposed for a FG beam and then the higher-order theory is established, including the equilibrium equations and the corresponding boundary conditions. In Section 6, typical FG beam problems are analytically solved by using the present modified high-order theory and then discussed as compared with traditional beam theories. The concluding remarks are finally made in Section 7.

## 2. Description of a FG beam problem

### 2.1. The coordinate system

To focus our attention on a FG beam, a straight beam structure with rectangular cross section and unit width is considered, as shown in Fig. 1. From the viewpoint of elasticity, this is a plane stress problem in the $x$-$z$ plane. Though the coordinate system can be set at one's discretion, the origin $z = 0$ is located in this context at the centroid of cross section for ease of formulations (see Section 3.2). That is, we have

$$\int_A z dA = 0 \tag{1}$$

where $A$ is the area of cross section.

Thus, for the current rectangular cross section, the bottom surface and the top surface can be denoted by $z = -h/2$ and $z = h/2$, respectively.

It should be noted that, as a FG beam structure concerning in this paper, elastic constants may be a smooth function expressed by $E(z)$ and $G(z)$ (Aydogdu and Taskin, 2007; Benatta et al., 2009; Ben-Oumrane et al., 2009; Thai and Vo, 2012).

### 2.2. The generalized displacements of a FG beam

Regarding this beam structure as a plane stress problem, the displacements to be solved are axial displacement $u_x(x, z)$ in the length direction and transverse displacement $u_z(x, z)$ in the thickness direction. Analogously, from the viewpoint of beam, the generalized displacements can be defined in the average sense over the cross section of the FG beam.

**Definition 1.** Based on the transverse displacement $u_z(x, z)$, deflection $w(x)$ of a FG beam is defined as (Reddy, 2011; Zhang, 2013a, 2013b; Şimşek, 2010; Duan and Li, 2015; Geng et al., 2017)

$$w(x) = \int_A E(z)\cdot u_z(x, z)dA/B_0 \tag{2}$$

where

$$B_0 = \int_A E(z)dA \neq 0 \tag{3}$$

As will be seen in Section 4.2, $B_0$ can be physically interpreted as the tensile rigidity of a FG beam.

**Definition 2.** Based on the axial displacement $u_x(x, z)$, rotation $\phi(x)$ of cross section about the width direction $y$ for a FG beam is defined as (Duan and Li, 2015; Cowper, 1966; Murthy, 1981)

$$\phi(x) = \frac{1}{B_2}\int_A E(z)(z - z_c)\cdot u_x(x, z)dA \tag{4}$$

where

$$B_2 = \int_A E(z)\cdot(z - z_c)^2 dA \neq 0 \tag{5}$$

As will be seen in Section 4.2, $B_2$ can be physically interpreted as the flexural rigidity of a FG beam.

In Eqs. (4) and (5), $z_c$ signifies the reference point [1] over the cross section of a FG beam in defining rotation (hence moment later).

**Definition 3.** Based on the axial displacement $u_x(x, z)$, stretch of beam in the length direction $x$ is defined as (Pradhan and Chakraverty, 2013; Timoshenko, 1921)

$$\bar{u}(x) = \int_A E(z)\cdot u_x(x, z)\ dA/B_0 \tag{6}$$

## 3. The deformation mode of a FG beam

In this section, the deformation mode is studied for a FG beam.

### 3.1. Assumptions and the condition in the beam problem

For a beam problem, following two assumptions are usually adopted.

**Assumption 1.** The transverse normal stress $\sigma_z$ of beam is too small to be ignored (Frikha et al., 2016; Aydogdu, 2009; Şimşek, 2010; Davalos et al., 1994). Thus, we have the reduced Hook's law over the cross section of a FG beam as[2]

[1] Only if a reference point is introduced, the frame-indifferent rotation (hence moment) is defined and therefore physically objective.

[2] In fact, Eq. (7) is directly assumed when studying curved beams (e.g. see (Kapania and Li, 2003a; Kapania and Li, 2003b; Simo et al., 1995; Simo and Vu-Quoc, 1991)).