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A comparative analysis of two formulations for non linear hardening plasticity models: Application to shakedown analysis

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ABSTRACT

The paper is mainly devoted to a comparative analysis of two plasticity models with a nonlinear hardening law, namely the Armstrong and Frederick model, and a recent modification proposed in literature in the framework of Generalized Standard Materials (GSM). We first provide a detailed mathematical analysis of the two models by appropriately resorting to the bipotential theory. This delivers for the GSM model a closed form expression of a bipotential. Moreover, it is demonstrated for the first time that the Armstrong and Frederick model does not admit a convex potential; this result confirms the necessary requirement of a non associated framework for this model. Then, for the modified model, making use of the above bipotential-based tools, we carry out a shakedown analysis of a thin walled tube under constant tension and alternating cyclic torsion. The accuracy of the obtained results is checked by comparing them to data obtained by numerical solving the corresponding shakedown bounds problems. Finally, the predictions of the two models are compared and illustrated their significant differences.

1. Introduction

The modelling of inelastic responses of engineering materials usually assumes Generalized Standard Materials (GSM) for which only a yield function playing also the role of a plastic potential is required. However, for materials exhibiting nonlinear kinematic hardening, the rigorous framework of GSM models appears to be inappropriate.[1] Concerning metal plasticity, a widely considered model with nonlinear hardening is that introduced by Armstrong and Frederick (1966) (see also Frederick and Armstrong, 2007; Chaboche, 1989, 1991; Lemaitre and Chaboche, 1990). This models allows to describe the salient features of metallic materials under complex multiaxial and cyclic loadings (see for instance Jiang and Kurath, 1996; Lubarda and Benson, 2002; Chelminski, 2003; Mahbadi and Eslami, 2006; Jiang and Zhang, 2008; De Angelis and Taylor, 2014). Owing to the presence of a recall term in the nonlinear kinematic rule, the flow rule does not comply with the normality law.

An alternative formulation to the Armstrong and Frederick model, based on a concept called *generalized plasticity* by Lubliner (1991), has been introduced and applied in Auricchio and Taylor (1995), in which a comparative analysis of the predictions of the two models has been performed. It has been shown that the non linear kinematic hardening model is computationally more expensive than the generalized plasticity model. Another point of view comes from studies by Erlicher and Point (see Erlicher and Point, 2006; Point and Erlicher, 2013b, 2013a). Indeed, based on endochronic theory (see for instance Valanis and Wu, 1975; Bazant, 1978), these authors proposed a pseudo-potential of dissipation within the standard theoretical framework of thermomechanics. Other several plasticity models, with non linear kinematic hardening rule, are shown to fulfil a generalized normality assumption. For instance, Francfort and Stefanelli (2013) have chosen to use the normality rule with generalized variables. They have introduced a coupled pseudo-potential of dissipation depending both on strain rate and stress; however it can be observed that such pseudo-potential is non convex regarding one of its variables. Finally, still about the nonlinear kinematic rule, Besson et al. (2010) (see also Besson et al., 2001) adopt the formalism of the generalized standard model using a new form of the plastic criterion, initially published by Ladevèze and Rougée (1985). Applications of such model have been presented in Gourgues and Andrieu (2003). All

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[1] This has been also recognized long time ago for frictional cohesive geomaterials (see for instance Maier and Hueckel, 1979).

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the above researches use various approaches to retrieve the Generalized Standard Material framework in order to overcome well-known computational difficulties inherent to the Armstrong and Frederick model.

In contrast, the Implicit Standard Materials (ISM), introduced by de Saxcé in de Saxcé (1992), de Saxcé in Maier and Weichert (2002) offer a suitable framework which allows to recover the flow rule normality in a weak form of an implicit relation. This concept of ISM, described by using the bipotential theory, has been successfully applied to the Armstrong and Frederick model, especially for shakedown studies (Bouby et al., 2006, 2009).

The objectives of the present study, mainly based on the use of a bipotential approach which will be presented in the following, are: i) to construct the bipotential for the Besson et al. model; ii) to provide a deeper mathematical insight for the Armstrong and Frederick model; in particular it will be demonstrated that the Armstrong and Frederick model does not admit a potential; iii) to perform a detailed and comparative analysis of the two models. To this end we will perform the study of shakedown of a thin walled tube under cyclic loading (constant tension and alternating cyclic torsion). It is convenient to recall that in the context of non-associative rules, the earlier extension of the shakedown criteria, built on the concept of reduced elastic domain and the description by a plastic potential distinct from the yield function, is due to Maier (1969). Pycko and Maier (1995) and Corigliano et al. (1995) have proposed an extension to elastic–plastic materials and Nayroles and Weichert (1993) the elastic sanctuary concept. In footsteps of these works, most of studies on non linear kinematic hardening are restricted to Generalized Standard Materials using a plastic potential different from the yield function and/or two-surfaces formulations (see e.g. Nguyen, 2003; Pham, 2007, 2008; Simon, 2013). Finally, note that some shakedown studies have been successfully done using the non linear kinematic hardening rule introduced by Armstrong and Frederick (see Bodovillé and de Saxcé, 2001; Bouby et al., 2006, 2009).

The paper is organised as follows. Section 2 is devoted to a brief recall of the Armstrong Frederick model describing the non linear kinematic hardening plasticity. In Section 3, after recalling the bipotential corresponding to the Armstrong–Frederick model, we proposed a bipotential based reformulation of the generalized standard modification of the non linear hardening plasticity model proposed by Besson et al. (2010). A rigorous proof that the Armstrong–Frederick model does not admit a convex potential is provided in Section 4. A fully shakedown analysis of a thin walled tube under a cyclic loading with Besson's model is performed in Section 5. Finally, we compare the shakedown loads predicted by the two models in Section 6. The results are discussed in order to illustrate the fundamentals differences between the two models.

2. Brief recall of the two studied models for non linear hardening plasticity

2.1. The Armstrong Frederick model for non linear hardening plasticity

A realistic representation of the cyclic plasticity of metals is given by the so-called non linear kinematical hardening rules. A simple and efficient one was proposed by Armstrong and Frederick (1966) (see also Frederick and Armstrong, 2007) and was popularized by Lemaitre and Chaboche (1990). We denote E_d the linear space of traceless symmetric tensors of rank 2 on $\mathbb{R}^3$. The primal variable $\dot{\kappa} = (\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}, -\dot{p})$ gathers the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p \in E_d$, the kinematical hardening variables $\dot{\boldsymbol{\alpha}} \in E_d$ and the isotropic one $\dot{p} \in \mathbb{R}$. The dual variable is $\pi = (\boldsymbol{\sigma}', \boldsymbol{X}, R)$ compound of the deviator of the Cauchy stresses $\sigma' \in E_d$, the back stresses $\boldsymbol{X} \in E_d$ and the current yield stress $R \in \mathbb{R}$. The plastic yield function is defined as:

$$f(\boldsymbol{\sigma}', \boldsymbol{X}, R) = \sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X}) - R \leq 0 \tag{1}$$

Equivalently, the elastic domain is described by the closed convex cone

$$K = \left\{ (\boldsymbol{\sigma}', \boldsymbol{X}, R) \in E_d \times E_d \times \mathbb{R} \quad \text{s.t.} \quad \sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X}) \leq R \right\}$$

with $\sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X}) = \sqrt{3/2(\boldsymbol{\sigma}' - \boldsymbol{X}) : (\boldsymbol{\sigma}' - \boldsymbol{X})}$.

The flow rule is given by the classical normality law:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{p} \frac{\partial f}{\partial \boldsymbol{\sigma}'} = \frac{3}{2} \dot{p} \frac{\boldsymbol{\sigma}' - \boldsymbol{X}}{\sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X})} \tag{2}$$

The non linear kinematical hardening rule is written:

$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\varepsilon}}^p - \frac{3}{2} \frac{\gamma \boldsymbol{X}}{C} \dot{p} = \dot{\boldsymbol{\varepsilon}}^p - \frac{3}{2} \frac{\boldsymbol{X}}{X_\infty} \dot{p} \tag{3}$$

and the back stresses are linearly dependent on the kinematic variables through

$$\boldsymbol{X} = \frac{2}{3} C \boldsymbol{\alpha}$$

where γ and C are material dependent constants and $X_\infty = C/\gamma$. Note that due to the presence of the recall term, called dynamic recovery, $\dot{\boldsymbol{\alpha}}$ differs from $-\partial f / \partial \boldsymbol{X} = \partial f / \partial \boldsymbol{\sigma} = \dot{\boldsymbol{\varepsilon}}^p$ which obviously proves that the non linear hardening law does not derive from the normality rule.

Moreover, it is worth while remarking that the values of $\sigma_{eq}(\boldsymbol{X})$ are asymptotically bounded. It can be seen easily by remarking that in the neighbour of the asymptotic surface, $\dot{\boldsymbol{\alpha}}$ approaches zero and then $\dot{\boldsymbol{X}}$ so is. On this surface, $\dot{\boldsymbol{\alpha}} = \boldsymbol{0}$ and owing to the non linear hardening rule (3) and the flow one (2), it holds

$$\frac{\boldsymbol{X}}{X_\infty} = \frac{2}{3} \frac{\dot{\boldsymbol{\varepsilon}}^p}{\dot{p}} = \frac{\boldsymbol{\sigma}' - \boldsymbol{X}}{\sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X})}$$

from which we deduce the equation of the asymptotic surface

$$\sigma_{eq}(\boldsymbol{X}) = X_\infty$$

For a given value σ_y of the current yield stress R, this defines the admissibility domain of Armstrong–Frederick model

$$K_{AF} = \left\{ \boldsymbol{\pi} = (\boldsymbol{\sigma}', \boldsymbol{X}) \quad \text{s.t.} \quad \sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X}) \leq \sigma_y \quad \text{and} \quad \sigma_{eq}(\boldsymbol{X}) \leq X_\infty \right\}$$

2.2. A generalized standard modification of the non linear hardening plasticity model proposed by Besson et al. (2010)

In order to overcome some well-known difficulties related to the non standard character of the hardening rule, Besson et al. (2010) proposed to introduce a new yield function allowing to recover (3) by means of a normality rule. The proposed yield function takes the form

$$f_B(\boldsymbol{\sigma}', \boldsymbol{X}) = \sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X}) + \frac{1}{2X_\infty} \sigma_{eq}^2(\boldsymbol{X}) - \sigma_y \leq 0 \tag{4}$$

from which the elastic domain reads

$$K_B = \left\{ \pi = (\boldsymbol{\sigma}', \boldsymbol{X}) \text{ such that } \sigma_{eq}(\boldsymbol{\sigma}' - \boldsymbol{X}) + \frac{1}{2X_\infty} \sigma_{eq}^2(\boldsymbol{X}) - \sigma_y \leq 0 \right\} \tag{5}$$