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# Free vibration of an isotropic elastic skewed parallelepiped – A closed form study

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ABSTRACT

The free-vibration of three-dimensional non-rectangular parallelepiped is studied, aiming at providing closed form expressions for the natural frequencies by means of a systematic approximation. To this end, a kinematic approximation is formulated based on Taylor's multivariable expansion, and constitutive relations for internal forces follow from analytical integration of the weak form. Based on this approach we investigate three general classes of parallelepipeds, namely a cube, a rectangular brick, and a skewed rhombohedron. Our second order closed form solution for the cube significantly improves currently available analytical approximations derived by Cosserat point theory. The improvement is not only in accuracy, but also in the ability to provide a richer response spectrum and to capture the lowest frequencies. In addition, based on a fourth order approximation, we derive a simple explicit expression for the fundamental (lowest) frequency covering the entire range of Poisson's ratios with high accuracy – less than 1.6% error compared to FE results. In addition, we obtain, for the first time, closed form expressions, based on a second order approximation, for the fundamental frequencies of rectangular bricks and of skewed rhombohedra. These solutions cover the entire range of aspect ratios, from thin plates through a cube to slender beams, and the entire range of skew angles.



## 1. Introduction

The parallelepiped is a fundamental structural building block in a wide range of engineering applications and constructions, ranging from long-span highway bridges to structural slabs, decks and panels. In addition, beams and plates, which are most abundant in structures, are a special case of the 3-D parallelepiped. Thus, 3-D analysis of parallelepipeds can be useful to define the range of applicability of simpler theories. Indeed, many approximate solutions were formulated for particular (simple) parallelepipeds. For example, if one of the dimensions of the parallelepiped is small relative to the other two then it degenerates to a thin plate, whereas if two of its dimensions are small relative to the third then it degenerates to a thin beam. The free vibrations of one dimensional beams (Hutchinson, 2001; Kaza and Kielb, 1984; Love, 2013; Rao and Carnegie, 1973; Strutt et al., 1945; Subrahmanyam and Kaza, 1985; Timoshenko, 1921) and of 2-D plates (Dokainish and Rawtani, 1969; Gill and Ucmaklioglu, 1979; Leissa et al., 1981; McGee and Giaimo, 1992; Olson and Lindberg, 1971; Petricone and Sisto, 1971; Walker, 1978) have been studied extensively. The problem of a vibrating 3-D parallelepiped, either rectangular or not, is much more difficult.

In the last several decades there has been an effort to provide 3-D elasticity solutions for the free vibration of prisms and parallelepipeds. Vast majority of these works have utilized various techniques to obtain *numerical results*. For example, a series expansion method was applied to analyze the 3-D vibration of a free rectangular parallelepiped in (Fromme and Leissa, 1970; Hutchinson and Zillmer, 1983). Later (Leissa and Zhang, 1983; Liew et al., 1995a), have used generalized orthogonal polynomials as trial functions for the same problem. In (Lim, 1999), the vibration of a rectangular parallelepiped was studied by neglecting transverse normal stresses. The study in (Zhou and McGee III, 2013) offers a comprehensive numerical study of 3-D vibration solutions for elastic skew prisms. Simple algebraic polynomials, algebraic-trigonometric polynomials, and Gram-Schmidt orthogonal polynomials have also been used in (Fromme and Leissa, 1970; Hutchinson, 1981; Hutchinson and Zillmer, 1983; Leissa et al., 1981; Leissa and Zhang, 1983; Liew and Hung, 1995; Liew et al., 1994, 1993, 1995a,b; Lim, 1999; McGee III and Kim, 2010;

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McGee and Giaimo, 1992; So and Leissa, 1997). Vibration of non-rectangular thick plates was considered in (McGee et al., 1999). These days, the natural frequencies of a parallelepiped can be calculated with standard commercial finite-element software. However, these solutions neither provide closed form expressions nor analytical insights. The main purpose of the current contribution is to provide closed form expressions by means of a systematic approximation. These solutions cover the entire range of practical parameters, and their strength rests in their ability to provide quantitative intuition that is applicable for a wide range of structural elements. The approach presented here can also be applied to other 3-D geometries, emphasizing its practical importance.

Only a few studies applied structural theories that provide closed form solutions/approximations for 3-D elasticity problems. These models take advantage of particular constitutive relations, justified by simplifying assumptions on the structural behavior, that lead to closed form equations of motion in terms of the degrees of freedom and also material and geometrical constants. The structural approaches which can be applied to 3-D problems are roughly divided to two main classes: Pseudo Rigid Body and Cosserat Point (e.g. Antman, 2005; Casey, 2006; Naghdi, 1972; Rubin, 1987; Rubin et al., 2002). Cosserat Point method defines kinematic approximations in terms of directors, and also imposes restrictions on the strain energy function. In particular, the strain energy is separated into two parts; one uses average measures of deformation and the other is restricted to admit exact solutions, such as simple torsion and pure bending, in an average sense. Cosserat point analytically satisfies Patch test, which ensures convergence when the structure dimensions (size) tend to zero, see for example (Jabareen et al., 2012; Jabareen and Rubin, 2010b, 2008b, 2010c). Pseudo-Rigid body is a Cosserat-like approach that enables closed form model for 3-D elastic solids of finite size. In addition to rigid body motion, it allows homogeneous deformation. Papadopoulos (Papadopoulos, 2001) has developed a second order theory of a pseudo-rigid body which has 30 degrees of freedom. It seems that the establishment of a pseudo-rigid body model on the basis of continuum mechanics is a delicate and unresolved issue (see Casey, 2006; Cohen and Muncaster, 1984; Steigmann, 2006). The pseudo-rigid body method is mainly used to simulate dynamics of multi-body systems which involve deformable solids with boundary interactions, such as contact (Cohen and Macsithigh, 1991; Cohen and Sun, 1988; Kanso and Papadopoulos, 2004; Muncaster, 1984; Slawianowski, 1982; Solberg and Papadopoulos, 1999, 2000; Zienkiewicz and Taylor, 2005).

The present study follows the basic guidelines of (Hanukah and Goldshtein, 2012) to formulate the governing equations of motion for a 3-D non-rectangular parallelepiped. In particular, the 3-D elasticity problem is converted into a set of closed form non-linear ODEs. Kinematic approximation is systematically derived by means of Taylor's multivariable expansion. Constitutive relations for internal forces follow from analytical integration of the weak form. The following analytical integration leads to closed-form expressions for internal forces and the stiffness matrix. Analytic linearization of the system enables us to derive simple closed form expressions for resonant frequencies in terms of geometric and material constants.

We note that the weak formulation we adopt is closely related to the finite element method (FEM) (Wriggers, 2008), and in particular to a p-type FEM with polynomial shape functions (Yosibash, 2012). However, while FEM expresses the kinematic approximation in terms of nodes, herein the kinematic approximation is rigorously expressed in terms of internal degrees of freedom which are consistent with a systematic approximation. We also note that closed form expressions for the internal forces and stiffness matrix have been previously derived based on FEM for linear elastic problems (e.g. Lee and Hobbs, 1998; McCaslin et al., 2012; Shiakolas et al., 1994, 1992). These were developed in order to reduce the numerical effort required by FEM. In the present study, we use exact analytical integration in order to write an explicit formulation of the governing equations and to obtain analytical solutions.

We use our approach to study three classes of parallelepipeds: a cube, a rectangular brick, and a skewed rhombohedron. Currently, *the only study that provides closed form expressions for the natural frequencies of a cube is* (Rubin, 1986), which builds on a simple Cosserat point theory (Rubin, 1985). This formulation involves twelve degrees of freedom which can provide six non-trivial natural frequencies (excluding rigid body motion). Unfortunately, these frequencies are not necessarily the fundamental ones, and lower frequencies may exist. This Cosserat point theory has been generalized later to include 24 degrees of freedom (Jabareen and Rubin, 2008a, 2010a; Nadler and Rubin, 2003a, 2003b) in order to account for inhomogeneous deformations and to be implemented into finite elements. However, this theory was not applied to the free vibration problem, and cannot be systematically generalized to formulate a higher order theory (higher than second order). We find that our first order solution is identical to the one obtained in (Rubin, 1986), which only provides 6 nontrivial modes that are associated with only *two frequencies*. Our second order approximation significantly improves the accuracy of the solution. Further, it predicts 24 non-trivial modes with nine different frequencies. This is a significant improvement over the first order solution in terms of describing the response spectrum of the structure. Moreover, the solution obtained in (Rubin, 1986) is not able to capture the lowest frequencies. We also find that different modes may be of greater engineering importance, i.e. be associated with lower frequencies, depending on Poisson's ratio. This stems from the fact that frequencies associated with different modes have a different functional dependence on Poisson's ratio. Finally, we provide a highly accurate simple analytical expression for the fundamental (lowest) frequency based on a fourth order approximation. Next, we apply our approach to derive, for the first time, a closed form expression for the fundamental frequency of a rectangular brick of dimensions $H \times H \times \varepsilon H$ based on a second order approximation. Our solution provides reasonable accuracy while covering the entire range of Poisson's ratios and the entire range of $\varepsilon$, i.e thin plates ($\varepsilon \ll 1$), thick plates ($\varepsilon < 1$), cubes ($\varepsilon = 1$), short beams ($\varepsilon > 1$), and slender beams ($\varepsilon \gg 1$). Finally, we derive a closed form expression for the fundamental frequency of a skewed parallelepiped (rhombohedron with angle $\phi$). The solution is based on a second order approximation and covers the entire range of Poisson's ratios and skew angles, $\phi$.

The outline of the paper is as follows. Section 2 presents the main theoretical considerations and formulation of the free vibration problem. In Section 3, we apply the formulation from Section 2 to study the free vibrations of three classes of parallelepipeds: a cube, a rectangular brick, and a skewed rhombohedron. The accuracy of our closed-form expressions for the natural frequencies is examined by a comparison with finite-element simulations. Summary and main conclusions are presented in Section 4.

## 2. Theoretical considerations

Consider a three dimensional body occupying a finite volume in Euclidean space, which has a parallelepiped shape in its reference configuration, made of an isotropic and linear-elastic material, free of gravitational body forces or tractions (see Fig. 1). First, the basic equations of linear elasticity are recalled. Balance of linear and angular momentum in absence of body forces