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On the angular velocity of a rigid body: Matrix and vector representations

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A R T I C L E I N F O

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A B S T R A C T

The angular velocity is an important property associated with the velocity state of moving rigid bodies. Unlike the velocity vector of a point, angular velocity vector is not in general equal to the time derivative of any single vector. Hence a unified, simple and comprehensible treatment of the subject may benefit the velocity analysis of complex multibody systems. This paper contributes with a new point of view of *matrix and vector* representations of angular velocity from the very foundations of classical kinematics of rigid bodies. This contribution was given a systematic, integrated and unified treatment, thus allowing the derivations to be based upon quantities which are expressed in terms of geometric objects (vectors) and geometric operations (vector addition, dot, and cross product). As a result, the approach leads naturally to simple and particularly useful expressions for the angular velocity vector, which allow a readily extension to three important representations involving the position and velocity of three noncollinear points pertaining to a moving rigid body.

1. Introduction

Angular velocity has been one of the most fundamental and most misunderstood quantities in kinematics. A possible reason for this misunderstanding is that there is not a complete analogy between *translational velocity* and *angular velocity*.[1] Hence the way one arrives at the angular velocity vector is completely different from the derivation of the expression for translational velocity, or the rate of change of any defined vector (Baruh, 1999).

One of the possible ways to arrive to the angular velocity vector ω is through what is known as *angular velocity matrix*, $\mathbf{\Omega}$. The general expression that defines the angular velocity matrix is $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$, which provides frequently used relationships between the time derivative, $\dot{\mathbf{R}}$, of a transformation matrix $\mathbf{R}$—a *proper orthogonal* matrix—and angular velocity vector ω of a body (Haug, 1989). From here, it should be clear the important relation that exists between the angular velocity matrix $\mathbf{\Omega}$ and the rotation matrix $\mathbf{R}$. Moreover, the rearranged form $\dot{\mathbf{R}} = \mathbf{\Omega}R$ is known as *Poisson's equation* (Zhuravlev, 2005; Sinclair and Hurtado, 2005; Amirouche, 2006).

A number of derivations (Amirouche, 2006; Angeles, 1982; Nikravesh, 1988; Haug, 1992; Bottema and Roth, 1990; Beggs, 1983; Barford, 1973; Laub and Shiflett, 1983; Fenton and Willgoss, 1990; Angeles, 2007; Pfeiffer, 2008; McCarthy and Soh, 2011; Jazar, 2011; Trindade and Sampaio, 2000) for the angular velocity matrix, $\mathbf{\Omega}$, and also for the angular velocity vector, ω—see for example (Kane and Levinson, 1985; Huston, 1990), among many others—have been reported in literature during several decades. These derivations have much in common and, to a greater or lesser extent, all of them follow a general pattern: a preconceived definition of the angular velocity. Thus, the conception of a physical phenomenon like angular velocity is forced to be born under the shadow of a relatively abstract mathematical apparatus, instead of preserving the natural features associated with the rotational motion of a rigid body undergoing an arbitrary spatial motion. As a consequence, the physical significance of the rotational motion is not always well understood. Hence the motivation of devising a more comprehensible derivation of the angular velocity—combining both representations[2] (matrix and vector) to form a whole—where the emphasis is put on describing the rotational phenomenon in a way that accords directly with physical insight.

At this point it is fair to recognize that all the concepts shown in Amirouche (2006), Angeles (1982), Nikravesh (1988), Haug (1992), Bottema and Roth (1990), Beggs (1983), Barford (1973), Laub and

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[1] It is important to realize that while angular velocity is a rigid body property, translational velocity depends on the chosen point.

[2] It should be noted that both, the angular velocity matrix and the angular velocity vector, admit several representations.

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Shiflett (1983), Fenton and Willgoss (1990), Angeles (2007), Pfeiffer (2008), McCarthy and Soh (2011), Jazar (2011), Trindade and Sampaio (2000), Kane and Levinson (1985) and Huston (1990) are clearly related, but distinct, to similar derivations set forth in this contribution. The main innovation presented is that, given the particular nature of our approach, it systematically groups a number of desirable features which allow a simpler and better way of conceiving and understanding the rotational motion of a rigid body, thus resulting in several advantages: (a) it is integral, since it considers the angular velocity concept as a whole: matrix and vector representations together, (b) contrary to most available results, it provides a detailed derivation based in vector quantities—without a preconceived definition of the angular velocity—, thus leading to a clear understanding of the physical features associated with a representation of the angular velocity matrix of the form $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$, (c) it leads naturally to simple and particularly useful expressions for the angular velocity vector, ω, and, (d) it allows a readily extension of the results to three useful and important representations of the angular velocity vector, which involve the position and velocity of three noncollinear points pertaining to a moving rigid body.

Finally, it is important to remark that, although the emphasis has been put on the rotational phenomenon, the multi-rigid-body system under analysis may undergo an arbitrary spatial motion.

2. Kinematic description of a moving rigid body

A rigid body that undergoes a spatial displacement may be described by the position vectors, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, of three noncollinear points pertaining to the body, see Fig. 1.

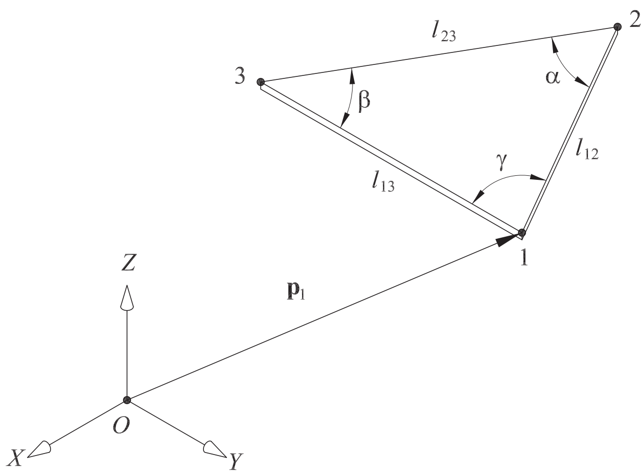

Fig. 1. A three-point rigid system representing a moving rigid body.

On one hand, since lines l_{12}, l_{13} and l_{23} belong to a rigid body, vectors $\mathbf{p}_2-\mathbf{p}_1$, $\mathbf{p}_3-\mathbf{p}_1$ and $\mathbf{p}_3-\mathbf{p}_2$ must remain of the same magnitude throughout a rigid-body motion. On the other hand, angles α, β and γ must be preserved throughout the motion as well. Thus, these three points must obey the following constraint equations:

$$(\mathbf{p}_3-\mathbf{p}_1)\cdot(\mathbf{p}_3-\mathbf{p}_1) = l_{13}^2 \tag{1}$$

$$(\mathbf{p}_2-\mathbf{p}_1)\cdot(\mathbf{p}_2-\mathbf{p}_1) = l_{12}^2 \tag{2}$$

$$(\mathbf{p}_3-\mathbf{p}_2)\cdot(\mathbf{p}_3-\mathbf{p}_2) = l_{23}^2 \tag{3}$$

$$(\mathbf{p}_2-\mathbf{p}_1)\cdot(\mathbf{p}_3-\mathbf{p}_1) = l_{12}l_{13}\cos\gamma \tag{4}$$

$$(\mathbf{p}_1-\mathbf{p}_2)\cdot(\mathbf{p}_3-\mathbf{p}_2) = l_{12}l_{23}\cos\alpha \tag{5}$$

$$(\mathbf{p}_1-\mathbf{p}_3)\cdot(\mathbf{p}_2-\mathbf{p}_3) = l_{13}l_{23}\cos\beta \tag{6}$$

which are usually known as *stiffness conditions* of a rigid body.

2.1. Pose of a moving rigid body

The *pose* of a rigid body requires the knowledge of the location of one point of the body and also, the attitude of the body. The location of a point is usually represented by a position vector and, the attitude of the body may be described in terms of the orientation of a frame attached to the body. Such requirements may be satisfied in terms of the position vectors of the three noncollinear points shown in Fig. 1. To this end, we define the following unit vectors:

$$\mathbf{u} \equiv \frac{\mathbf{p}_2-\mathbf{p}_1}{\sqrt{(\mathbf{p}_2-\mathbf{p}_1)\cdot(\mathbf{p}_2-\mathbf{p}_1)}} = \frac{\mathbf{p}_2-\mathbf{p}_1}{l_{21}} \tag{7}$$

$$\mathbf{m} \equiv \frac{\mathbf{p}_3-\mathbf{p}_1}{\sqrt{(\mathbf{p}_3-\mathbf{p}_1)\cdot(\mathbf{p}_3-\mathbf{p}_1)}} = \frac{\mathbf{p}_3-\mathbf{p}_1}{l_{31}} \tag{8}$$

$$\mathbf{v} \equiv \frac{\mathbf{m}-(\mathbf{m}\cdot\mathbf{u})\mathbf{u}}{\lambda}, \quad \lambda \equiv \sqrt{\{\mathbf{m}-(\mathbf{m}\cdot\mathbf{u})\mathbf{u}\}\cdot\{\mathbf{m}-(\mathbf{m}\cdot\mathbf{u})\mathbf{u}\}} \tag{9}$$

$$\mathbf{w} \equiv \mathbf{u}\times\mathbf{v} \tag{10}$$

In this way, the pose of the body may be represented by position vector $\mathbf{p}_1$ and the unit vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, which constitute a moving frame *UVW* (a right-handed orthonormal coordinate system), as it is shown in Fig. 2.

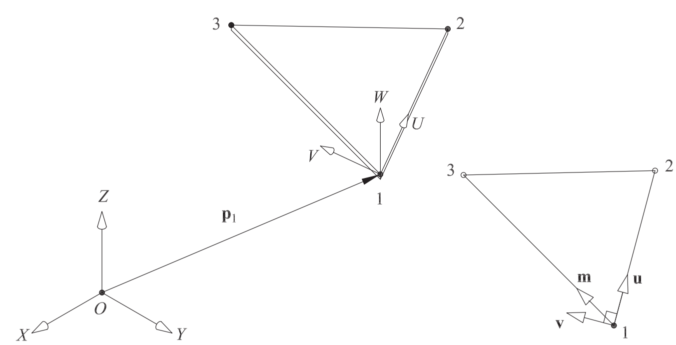

Fig. 2. A moving frame *UVW* attached to a moving rigid body.

It is important to remark that unit vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ have been completely defined in terms of the position vectors, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, of three noncollinear points pertaining to a moving rigid body. Hence the pose of the body under analysis has been fully characterized in a simple and convenient way.

2.2. Rotation matrix

The attitude of the body under analysis may be represented by a suitable rotation matrix that serve to describe the orientation of the moving frame $\mathcal{M} : UVW$ with respect to a fixed frame $\mathcal{F} : XYZ$,[3] see Fig. 3. This rotation matrix is computed next.

[3] Both frames, $\mathcal{M} : UVW$ and $\mathcal{F} : XYZ$, are right-handed orthonormal coordinate systems.