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Reconstruction of an elliptical inclusion in the inverse conductivity problem

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ABSTRACT

This study reports on a numerical investigation into the open problem of the unique reconstruction of an elliptical inclusion in the potential field from a single set of nontrivial Cauchy data. The investigation is based on approximating the potential fields of a composite material as a linear combination of fundamental solutions for the Laplace equation with sources shifted outside the solution domain and its boundary. The coefficients of these finite linear combinations are unknown along with the centre, the lengths of the semi-axes and the orientation of the sought ellipse. These are determined by minimizing the least-squares objective functional describing the gap between the given and computed data. The extension of the proposed technique for the reconstruction of two ellipses is also considered.

1. Introduction

One hundred years ago Johann Radon discovered the transform on which the principles of X-ray tomography are based. However, it took fifty years for its importance to be realized and acknowledged. The mathematical foundation of tomographic scanning was produced by A. Calderon in his seminal presentation in 1980. Since then, numerous breakthroughs have occurred on establishing the uniqueness of recovering the heterogeneous conductivity of a medium from the Dirichlet-to-Neumann boundary map culminating with the proof in two dimensions [2] for the unique recovery of L^∞-conductivities. However, one of the drawbacks of the Calderon formulation is that infinitely dimensional input data are required. Therefore, in order to render the formulation more practical, a series of papers was initiated by V. Isakov in the late eighties concerning the recovery of a piecewise conductivity from a finite set of Cauchy data [7,8]. This latter problem may be reformulated as a transmission problem for determining the interface between materials having different conductivities.

Convex or concave polygonal interfaces are uniquely identifiable from one or two sets of Cauchy data [4,26] but smooth surfaces are more difficult to investigate and, up to now, uniqueness with one set of Cauchy data is only known for circular or spherical interfaces [12,14]; also confirmed by stability estimates [16,27] and successful numerical reconstructions [15,21]. However, for other smooth shapes, e.g. ellipses, identification with one measurement is only known for perfectly conductive or insulated interfaces, i.e. piecewise extreme conductivities of ∞ or 0, [13]. Therefore, encouraged by some successful numerical investigations in which arbitrary smooth inclusions were recovered using either the boundary element method (BEM) [5,6] or the method of fundamental solutions (MFS) [17], see also [18,19], it is the purpose of this study to investigate the numerical identification of an elliptical interface from one measurement of Cauchy data in order to offer insight into the uniqueness of the yet unsolved inverse elliptical conductivity problem [9]. We note that the shape of an ellipse for an interface is typical for both damage and porosity geometries [3].

The paper is organized as follows. In Section 2 we provide the mathematical formulation of the inverse conductivity problem for identifying an elliptical inclusion from one Cauchy boundary data measurement. The approximation of the resulting transmission problem in a composite material using the MFS is presented in Section 3 and the resulting nonlinear minimization problem is described in Section 4. Several examples concerning the reconstruction of circular, elliptical and bi-elliptical inclusions are presented and discussed in Sections 5 and 6. Finally, in Section 7 we present some conclusions and ideas for future work.

2. Mathematical formulation

We consider the inverse conductivity problem of determining a piecewise constant isotropic conductivity $1+(\kappa-1)\mathcal{X}(D)$, where D is an unknown inclusion (in this paper an ellipse or a collection of ellipses) compactly contained in a given planar bounded domain $\Omega\subset\mathbb{R}^2$, where $\mathcal{X}(D)$ is the characteristic function of the domain D and $\kappa\neq 1$ is a given positive constant, from a single measurement of the current flux

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induced by a boundary potential prescribed on $\partial\Omega$ or vice versa. This inverse problem represents the mathematical formulation of the continuous model of electrical capacitance/impedance tomography. It can be recast as the following transmission problem governed by the Laplace equations:

$$\Delta u_1 = 0 \quad \text{in} \quad \Omega\backslash\overline{D}, \tag{2.1a}$$

$$\Delta u_2 = 0 \quad \text{in} \quad D, \tag{2.1b}$$

subject to the boundary conditions

$$u_1 = f \not\equiv \text{constant} \quad \text{on} \quad \partial\Omega, \tag{2.1c}$$

$$\frac{\partial u_1}{\partial n} = g \quad \text{on} \quad \partial\Omega, \tag{2.1d}$$

and the transmission perfect contact conditions

$$u_1 = u_2 \quad \text{on} \quad \partial D, \tag{2.1e}$$

$$\frac{\partial u_1}{\partial n^-} = -\kappa\frac{\partial u_2}{\partial n^+} \quad \text{on} \quad \partial D, \tag{2.1f}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected planar domain with smooth boundary $\partial\Omega$ and ∂D is the ellipse defined by

$$x = X + r(\vartheta)\cos\vartheta, \quad y = Y + r(\vartheta)\sin\vartheta, \quad \vartheta \in [0, 2\pi), \tag{2.1g}$$

and

$$r(\vartheta) = \frac{1}{\sqrt{\dfrac{\cos^2(\vartheta-\varphi)}{a^2} + \dfrac{\sin^2(\vartheta-\varphi)}{b^2}}}. \tag{2.1h}$$

In (2.1g) and (2.1h), (X, Y) is the centre of the ellipse, $2a$ and $2b$ are the lengths of the major and minor axes of the ellipse, respectively, and φ is the angle the major axis makes with the horizontal. Similar considerations can be made for an ellipsoid in three dimensions using spherical coordinates.

3. The method of fundamental solutions (MFS)

The MFS for the Laplace equation in a bounded domain may be viewed as a numerical discretization of a single-layer potential boundary integral representation in which the given boundary values and the sought solution are defined on different curves [11]. Consequently, a solution to the Laplace equation (2.1a) is given as a linear combination of fundamental solutions of the form

$$u_1(\boldsymbol{c}, \xi; \boldsymbol{x}) = \sum_{k=1}^{M+N} c_k\, G(\boldsymbol{x}, \xi_k), \quad \boldsymbol{x} \in \overline{\Omega}\backslash D, \tag{3.1}$$

where G is the fundamental solution of the two-dimensional Laplace equation, given by

$$G(\xi, \boldsymbol{x}) = -\frac{1}{2\pi}\log|\,\xi - \boldsymbol{x}\,|. \tag{3.2}$$

The sources $(\xi_k)_{k=\overline{1,M}}$ are located outside $\overline{\Omega}$, while the sources $(\xi_k)_{k=\overline{M+1,M+N}}$ are located in D. The geometry of the problem and the location of the source points are sketched in Fig. 1. More specifically, the sources $(\xi_k)_{k=\overline{1,M}}$ are located on a (moving) pseudo-boundary $\partial\Omega'$ similar to (dilation $\delta_1 > 0$) $\partial\Omega$ while the sources $(\xi_k)_{k=\overline{M+1,M+N}}$ are located on a (moving) pseudo-boundary ∂D^- similar to (contraction $\delta_2 > 0$) ∂D.

Similarly, we seek an approximation to the solution of the Laplace equation (2.1b) in the form

$$u_2(\boldsymbol{d}, \boldsymbol{\eta}; \boldsymbol{x}) = \sum_{k=1}^{N} d_k\, G(\boldsymbol{x}, \boldsymbol{\eta}_k), \quad \boldsymbol{x} \in \overline{D}, \tag{3.3}$$

where the sources $(\boldsymbol{\eta}_k)_{k=\overline{1,N}}$ are located outside $\overline{D}$ on a (moving) pseudo-boundary ∂D^+ similar to (dilation $\delta_3 > 0$) ∂D. The idea of using a fictitious moving pseudo-boundary in inverse geometric problems was first proposed in [19].

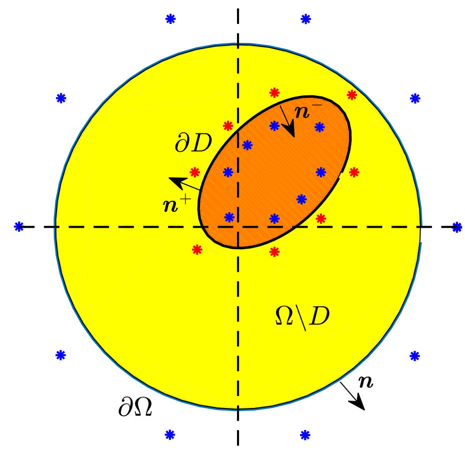

Fig. 1. Geometry of the problem. The asterisks (*) denote the source points located on fictitious pseudo-boundaries $\partial\Omega'$ (dilation of $\partial\Omega$), ∂D^- (contraction of ∂D) and ∂D^+ (dilation of ∂D).

Since we have $2M$ Cauchy boundary conditions (2.1c) and (2.1d) and $2N$ interface conditions (2.1e) and (2.1f) we have a total of $2M + 2N$ equations. The unknowns consist of the $M + N$ coefficients $(c_k)_{k=\overline{1,M+N}}$, the N coefficients $(d_k)_{k=\overline{1,N}}$, the centre (X, Y), the semi-axes of the ellipse a and b, the angle φ and the three dilation/contraction coefficients δ_1, δ_2, δ_3, yielding a total of $M + 2N + 8$ unknowns. In order to avoid an under-determined situation we require $M \geq 8$.

We next define the collocation points $(\boldsymbol{x}_\ell)_{\ell=\overline{1,M+N}}$, where $\boldsymbol{x}_\ell = (x_\ell, y_\ell)$, the sources $(\xi_k)_{k=\overline{1,M+N}}$, where $\xi_k = (\xi_k^x, \xi_k^y)$, and the sources $(\boldsymbol{\eta}_k)_{k=\overline{1,N}}$, where $\boldsymbol{\eta}_k = (\eta_k^x, \eta_k^y)$. Without loss of generality, we shall assume that the (known) fixed exterior boundary $\partial\Omega$ is a circle of radius R. As a result, the outer boundary collocation and source points are chosen as

$$\boldsymbol{x}_m = R(\cos\theta_m, \sin\theta_m), \quad m = \overline{1, M}, \tag{3.4}$$

$$\xi_m = \delta_1 R(\cos\theta_m, \sin\theta_m), \quad m = \overline{1, M}, \tag{3.5}$$

respectively, where $\theta_m = \frac{2\pi(m-1)}{M}$, $m = \overline{1, M}$, and the (unknown) parameter $\delta_1 \in (1, S_1)$ with $S_1 > 1$ prescribed.

We choose the inner boundary collocation and source points as

$$x_{M+n} = X + r(\vartheta_n)\cos\vartheta_n, \;\; y_{M+n} = Y + r(\vartheta_n)\sin\vartheta_n, \tag{3.6}$$

$$\xi^x_{M+n} = X + \delta_2\, r(\vartheta_n)\cos\vartheta_n, \;\; \xi^y_{M+n} = Y + \delta_2\, r(\vartheta_n)\sin\vartheta_n, \tag{3.7}$$

and

$$\eta^x_n = X + \delta_3\, r(\vartheta_n)\cos\vartheta_n, \;\; \eta^y_n = Y + \delta_3\, r(\vartheta_n)\sin\vartheta_n, \tag{3.8}$$

$n = \overline{1, N}$ where $\vartheta_n = \frac{2\pi(n-1)}{N}, n = \overline{1, N}$, and the (unknown) parameter $\delta_2 \in (S_2, 1)$ (with $0 < S_2 < 1$ prescribed) and the (unknown) parameter $\delta_3 \in (1, S_3)$ with $S_3 > 1$ prescribed.

4. Implementational details

The coefficients $(c_k)_{k=\overline{1,M+N}}$ in (3.1), the coefficients $(d_k)_{k=\overline{1,N}}$ in (3.3), the contraction coefficient δ_2 and the dilation coefficients δ_1, δ_3 in (3.5), (3.7), (3.8), the coordinates of the centre (X, Y), the half-lengths of the major and minor axes a and b in (2.1g) and the angle φ in (2.1g) can be determined by imposing the boundary conditions (2.1c) and (2.1d) and the transmission conditions (2.1e) and (2.1f) in a least-squares sense. This leads to the minimization of the functional

$$S(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{\delta}, \boldsymbol{C}, a, b, \varphi) := \sum_{j=1}^{M}\left[u_1(\boldsymbol{c}, \xi; \boldsymbol{x}_j) - f(\boldsymbol{x}_j)\right]^2 + \sum_{j=1}^{M}\left[\frac{\partial u_1}{\partial n}(\boldsymbol{c}, \xi; \boldsymbol{x}_j) - g(\boldsymbol{x}_j)\right]^2$$