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A closed-form solution for natural frequencies of thin-walled cylinders with clamped edges

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ABSTRACT

This paper presents an approximate closed-form solution for the free-vibration problem of thin-walled clamped–clamped cylinders. The used indefinite equations of motion are classic. They derive from Reissner's version of Love's theory, properly modified with Donnell's assumptions, but an innovative approach has been used to find the equations of natural frequencies, based on a solving technique similar to Rayleigh's method, on the Hamilton's principle and on a proper constructions of the eigenfuctions.

Thanks to the used approach, given the geometric and mechanical characteristics of the cylinder, the model provides the natural frequencies via a sequence of explicit algebraic equations; no complex numerical resolution, no iterative computation, no convergence analysis is needed.

The predictability of the model was checked both against FEM analysis results and versus experimental and numerical data of literature. These comparisons showed that the maximum error respect to the exact solutions is less than 10% for all the comparable mode shapes and less than 5%, on the safe side, respect to the experimental data for the lowest natural frequency.

There are no other models in the literature which are both accurate and easy to use. The accurate models require complex numerical techniques while the analytical models are not accurate enough. Therefore the advantage of this novel model respect to the others consists in a best balance between simplicity and accuracy; it is an ideal tool for engineers who design such shells structures.

1. Introduction

Structural elements similar to thin-walled cylinders are widely used in several engineering fields; for example, cylindrical shell-like structures exist in pipelines, submarine hulls, aircraft fuselages and missiles. During mechanical processing needed for their manufacture or during their normal use, these elements are often stressed by time-varying forces; consequently, there is a need to characterize the vibratory behaviour to optimise the design and the production process.

The present paper is composed of five sections and an appendix. This section provides a short historical review of the numerical and analytical models of free vibrations of thin elastic shells. Section 2 presents the differential equations of motion. In Section 3 and in Appendix the mathematical basis of the present model is outlined, and the key equations are derived. A detailed analysis of the results, together with several comparisons with other models and experimental data, is presented in Section 4, followed by conclusions in Section 5.

In the literature, there are several theories with various assumptions and simplifications about the vibrations of thin elastic shells; these theories typically are based on Love's indefinite equilibrium equations derived at the end of 19th century [1]. The research on this topic intensified during the 1960 s and 1970 s [2] and was further developed in the last two decades [3,4]. Over the years, linear models valid for small deformations were developed, along with non-linear models [5] also valid for large deformations.

In particular, the natural vibrations of thin-walled circular cylindrical shells were extensively analysed both from a theoretical point of view [6–13] and from an experimental point of view [6,7,11]; a recently published study aimed to adapt the classical theories to new applications based on carbon nanotubes [14]. However, due to the complexity of the problem, the exact solution of indefinite equations of motion only exists for circular cylindrical shells with two opposite shear diaphragm edges [12]. With other boundary conditions, the integration of these equations is generally performed with the aid of numerical methods; only in a few cases the solution has been found analytically, thanks to the introduction of special simplifying assumptions, but to the

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Nomenclature

a	mean radius of the cylinder [m]
A	amplitudes of the displacement functions [m]
D	bending rigidity of the thin wall [N m]
E	Young's modulus [N m^{-2}]
f	natural frequency [Hz]
$\mathbf{F}$	resultant force vectors per unit length [N m^{-1}]
G	shear modulus [N m^{-2}]
H	Hamiltonian action [J s]
h	wall thickness of the cylinder [m]
k	curvatures of the reference surface [m^{-1}]
K	extensional rigidity of the shell wall [N m^{-1}]
l	length of the cylinder [m]
L	Lagrangian function [J]
m	number of longitudinal half-waves
$\mathbf{M}$	resultant moment vectors per unit length [N]
M	moments, per unit length, acting on the infinitesimal element [N]
n	number of circumferential waves
N	forces, per unit length, acting on the infinitesimal element [N m^{-1}]
Q	transverse shear forces, per unit length, acting on the infinitesimal element [N m^{-1}]
$\mathbf{r}$	radial unit vector [m]
r	radial coordinate [m]
R_1, R_2, R_3	coefficients of the frequency equation
$\mathbf{s}$	circumferential unit vector [m]
s	circumferential coordinate [m]
t	time [s]
u	displacements of the reference surface [m]
W	virtual work [J]
$\mathbf{x}$	longitudinal unit vector [m]
x	longitudinal coordinate [m]
X	dimensionless longitudinal coordinate
α	model dimensionless parameter (see Eq. (26))
β	rotations of the tangents to the reference surface [rad]
γ	shear deformation of the reference surface
$\gamma(\zeta)$	shear deformation of a generic point
Δ	dimensionless frequency factor
ε	normal deformations of the reference surface
$\varepsilon(\zeta)$	normal deformations of a generic point
ζ	radial distance of a generic point from the reference surface [m]
η_1, η_2	model dimensionless parameter (see Eqs. (21) and (22))
θ	dimensionless circumferential coordinate [rad]
μ	model dimensionless parameter (see Eqs. (14) and (17))
ν	Poisson's ratio
ξ	model dimensionless parameter (see Eq. (20))
ρ	material density [kg m^{-3}]
τ	torsion of the reference surface [rad m^{-1}]
φ	model dimensionless parameter (see Eq. (26))
Ψ	model dimensionless parameter (see Eq. (12))
ω	circular frequency [rad s^{-1}]

Subscripts

s	circumferential direction
r	radial direction
x	longitudinal direction

detriment of accuracy or applicability domain. Arnold and Warburton [6] were among the first to study this type of problem; using the energy method and Timoshenko's relationships [15], they obtained a closed-form approximation of the natural frequencies for the case of simply supported edges. Koval and Cranch [7] studied the case of clamped–clamped edges using Donnell's equations [16] and provide an analytical solution as in this paper, but their model gets a limited applicability domain due to several oversimplifications. The same issue was addressed by Smith and Haft [8] using Flügge's equations [17] decoupled by Yu [18] but in this case as well, the problem was only solved numerically. Also Xuebin [19] used the Flügge's equations but introducing a new form of variables separation for arbitrary boundary conditions and applying the Newton–Raphson iteration method for the resolution of the frequency equation. Chung [20], using the Sanders' shell equations, obtained the expression of the frequency equation for any kind of boundary condition, but with the aid of iterative numerical method. Callahan and Baruh [9] obtained the natural frequencies analytically for several boundary conditions using Junger and Feit's equations [21]. However, the calculation was based on coefficients dependent on the constraints of geometry and material characteristics, which can be determined only numerically; therefore, this is not really a closed-form model. Wang and Lai [10] introduced a novel approach based on the wave theory and on the well-known Love's equations, which allowed them a closed-form resolution for different boundary conditions, clamped–clamped included as in this paper; however, the solution results inaccurate for the simpler mode shapes, as occurred for the Koval and Cranch [7] model. Pellicano [11] conducted both theoretical and experimental analyses on linear and nonlinear vibration based on the Sanders–Koiter theory [22–23] for different boundary conditions; in this case, the analysis was also performed using numerical resolution techniques. Recently, further approaches to the problem were developed: Xing et al. [12], working from the Donnell–Mushtari theory [24], resolved the problem for different boundary conditions via the variables separation method associated with the Newton iterative method; moreover, both Xie et al. [13] and Zhang et al. [25] analysed different boundary conditions using the Goldenveizer–Novozhilov theory [26] but with different numerical approaches, the former used the Haar wavelet numerical method, while the latter used the local adaptive differential quadrature method. Khalili et al. [27] presented a formulation of 3D refined higher-order shear deformation theory for the free vibration analysis of simply supported-simply supported and clamped–clamped cylindrical shells and the solutions are obtained using the Galerkin numerical method.

The literature review found no models for the free-vibration problem of clamped–clamped cylinders, which are both accurate and easy to use. The accurate models require complex numerical techniques while the analytical models are not accurate enough. The novel model presented here, in contrast, combines good accuracy with ease and speed of calculation: it carefully provides the natural frequencies via a simple sequence of explicit algebraic equations; no complex numerical resolution, no iterative computation, no convergence analysis is needed, unlike other models in the literature or FEM analysis.

The used indefinite motion equations were classic, but an innovative approach was used to find the equations of natural frequencies based on a solving technique similar to Rayleigh's