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How does axial prestretching change the mechanical response of nonlinearly elastic incompressible thin-walled tubes

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ABSTRACT

Human arteries provide an example of anisotropic, nonlinearly elastic, incompressible tubes. It is known that they operate *in situ* with significant axial prestretching. In ageing, this prestretching is successively relaxed due to arteriosclerosis. *Ex vivo* inflation-extension experiments have shown that axial prestretching is advantageous from the mechanical point of view, because it reduces the extent of the axial stress and strain that is experienced by arteries during the heart cycle. It has also recently been found that axial prestretch enhances circumferential deformations. Highly prestretched arteries exhibit higher circumferential stretches than their weakly prestretched counterparts, and this is advantageous when blood is transported to the periphery. However, this effect of axial prestretch on the mechanical response of a tube has until now been overlooked in the scientific literature. The objective of our study is to elucidate the physical cause of this phenomenon. An analytical model of an incompressible thin-walled closed tube was used to simulate the mechanical response of an initially prestretched tube to internal pressure. Four different situations were considered: (I) a hyperelastic material with a large strain stiffening property, (II) a neo-Hookean material, (III) a neo-Hookean material with small strains but large displacements (second order linear elasticity), and (IV) a neo-Hookean material with small strains and small displacements (first order linear elasticity). The results have shown that the positive effect of axial prestretch is not a property exclusively related to anisotropy. It has been proved that nonlinear effects are crucial. Nonlinear constitutive models depending on more than one parameter can both enhance and suppress the circumferential distensibility of the tube due to prestretching. However, a one-parameter neo-Hookean model showed only increased circumferential distensibility. A reduction in second order linear elasticity led to mechanical responses that exhibited only a slight effect of being prestretched. Total linearization proved that axial prestretch has a positive effect only to the point where deformed configuration and reference configuration are distinguished.

1. Introduction

An analytical model of a thin-walled tube based on the Laplace law is frequently used in physics and in the engineering sciences to obtain an elementary picture of a mechanical state. In biomechanics, nonlinearly elastic incompressible tubes are used to model arteries, veins, the oesophagus and other tubular organs [10,11,31,40]. The solutions that are obtained are usually considered to be first-order approximations, because the imposed assumptions of the model (the thickness-to-radius ratio, the residual stress and strain, the geometrical non-uniformity etc. [17,18,28]) are imperfectly satisfied. The simplicity of the thin-walled tube model might induce the impression that our knowledge of its mechanical response is exhaustive. In this study, however, we will show an example of a phenomenon that has been overlooked until now: the enhanced circumferential distensibility of a pressurized tube due to initial axial prestretching.

Human arteries *in situ* are significantly prestretched in the axial direction (this was probably first reported in the context of biomechanics by Fuchs in 1900 [7], as mentioned by Bergel [2]). This prestretching is observed during an autopsy as a retraction of the excised arterial segment [23–27], and the prestretch λ_{zz}^{ini} is defined as the ratio of the *in situ*-to-*ex situ* length of the segment. *Ex vivo* inflation-extension experiments have shown that axial prestretching is advantageous from the mechanical point of view, because it reduces the extent of the axial stress and strain that is experienced by arteries during the heart cycle [6]. In the optimal case of a young and healthy individual, there is a certain prestretch value at which the artery can be pressurized without a significant

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change to its length, so it can transmit a pressure pulse wave with negligible axial deformation [37,38,41].

However, recent studies by Horný et al. [23–26,28] have shown that ageing of the cardiovascular system is, besides general stiffening of elastic arteries, also manifested by a reduction of axial prestretch. Nevertheless, a detailed analysis of the constitutive behaviour of 17 human aortas suggested that aged aortas, although weakly prestretched, still can benefit from the remaining prestretch [28]. The decreasing of the prestretch is individual process similarly to (perhaps better to say as a consequence of) the progress of human ageing. A statistical variability reported in Horný et al. [28] implies that, for example, a 60-year-old man has the expected axial prestretch $\lambda_{zZ}{}^{ini}=1.08$ with a 95% confidence interval for a prediction $\lambda_{zZ}{}^{ini}\in[1.00;1.16]$. An analytical simulation of the inflation-extension response showed that, depending on the initial prestretch, the abdominal aorta of a 60-year-old man sustains the following changes in axial stretch $\lambda_{zZ}(P_{SYSTOLE})-\lambda_{zZ}(P_{DIASTOLE})=0.016$, 0.003, and 0.025 for $\lambda_{zZ}{}^{ini}=1.08$, 1.16, and 1.00, respectively [28]. The corresponding normalized changes in the axial Cauchy stress $(\sigma_{zz}(P_{SYSTOLE})-\sigma_{zz}(P_{DIASTOLE}))/\sigma_{zz}(P_{SYSTOLE})$ were 0.604 for expected prestretch $\lambda_{zZ}{}^{ini}=1.08$, 0.426 for the upper confidence limit of the prestretch $\lambda_{zZ}{}^{ini}=1.16$, and 0.769 for the lower limit $\lambda_{zZ}{}^{ini}=1.00$. This clearly demonstrates that, although axial prestretch declines (the expected prestretch of the abdominal aorta for a 20-year-old man is 1.34, with a 95% confidence interval for the prediction [1.24;1.43]), remaining prestretch still retains its biomechanical role: to minimize the axial stretch and stress variation.

Horný et al. [28] have moreover shown that axial prestretching also has a significant effect on the circumferential stretch variation $\lambda_{\theta\Theta}(P_{SYSTOLE})-\lambda_{\theta\Theta}(P_{DIASTOLE})$ exhibited during pressurization. Unlike the axial stretch and stress variations, which are minimized by prestretching, circumferential stretch variations were found to be increased by prestretching. For the same example as before of a 60-year-old man, the circumferential stretch variation $\lambda_{\theta\Theta}(P_{SYSTOLE})-\lambda_{\theta\Theta}(P_{DIASTOLE})$, which we will refer to here as distensibility, was computed to be 0.059 for $\lambda_{zZ}{}^{ini}=1.08$; 0.067 for $\lambda_{zZ}{}^{ini}=1.16$; and 0.056 for $\lambda_{zZ}{}^{ini}=1.00$. The study conducted by Horný et al. [28] revealed this phenomenon for all 17 investigated aortas. Higher axial prestretching induced inflation responses exhibiting higher circumferential distensibility of the tubes. This implies that the arterial physiology benefits in two ways from prestretching. The first way is from minimization of the axial stress and strain variation during the heart cycle, and the second way is from maximization of the circumferential distensibility during the cycle. Since arteries are conduits for the flowing blood, the higher distensibility of prestretched arteries means that they can accommodate a greater volume of blood at the same pressure than their non-prestretched counterparts. This leads us to regard the effect of axial prestretching as positive. To the best of our knowledge, this is the first time that such a conclusion on the effect on circumferential distensibility described in Horný et al. [28] has been presented in the literature.

In the authors' opinion, the positive effect of axial prestretching on circumferential distensibility is rather contra-intuitive at first sight, because we might expect a nonlinearly elastic tube to reach a stiffer state when pretension is applied. In their study, Horný et al. [28] hypothesized that anisotropy may be responsible for this phenomenon, because the elastic artery wall is reinforced by helically aligned collagen fibres [12,17,21,22] and Horný et al. [28] did indeed use an anisotropic constitutive model. However, they did not compare their results with the mechanical response of isotropic tubes, and anisotropy as a cause of the phenomenon remained only a hypothesis.

An objective of our paper is to show what physical mechanism is responsible for the enhancement the circumferential distensibility of an inflated tube. A bottom-up approach will be used to demonstrate what happens. The model of an incompressible nonlinearly elastic thin-walled tube will be simplified step-by-step from a material with exponential elastic potential at large strains to a linearly elastic material at small strains, and the cause of the enhanced circumferential distensibility will be made clear. We can state in advance that a problem formulated with large displacements but small strains for a linear material (second order linear elasticity) exhibits enhanced circumferential distensibility, and in the elementary linear elasticity of small displacements the model shows no positive effect of axial prestretch.

2. Methods

Two different analytical models were used in Horný et al. [28]. These were the thick-walled computational model, which is capable of accounting for residual strains, and the thin-walled model, which operates with averaged stresses acting on mid-surface of the tube. As is documented in that paper, the two models, although they differ numerically, give the same qualitative result – axial prestrain enhances circumferential distensibility. Since the effect of prestretching is captured by both models, in what follows, for the sake of simplicity and for clear and easy interpretation of the results, only the thin-walled model will be of our interest.

Consider a long thin-walled cylindrical tube with closed ends that, in the reference configuration, has middle radius R, thickness H, and length L. Assume that, during pressurization, the motion of the material particle located originally at (R, Θ, Z), which is sufficiently distant from ends, is described by the equations summarized as follows:

$$r=\lambda_{\theta\Theta}R,\quad h=\lambda_{rR},\quad z=\lambda_{zZ}Z,\quad \theta=\Theta \tag{1}$$

Here r denotes the deformed middle radius and h denotes thickness. Eq. (1) express the fact that the tube inflates and extends (or shortens) uniformly, and that it does not twist. The stretches λ_{kK} ($k=r$, θ, z; $K=R$, Θ, Z) are the components of the deformation gradient $\mathbf{F}$, $\mathbf{F}=diag[\lambda_{rR},\lambda_{\theta\Theta},\lambda_{zZ}]$. The right Cauchy–Green strain tensor $\mathbf{C}$ and Green–Lagrange strain tensor $\mathbf{E}$ can be computed as $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ and $\mathbf{E}=(1/2)(\mathbf{C}-\mathbf{I})$, where $\mathbf{I}$ is a second-order unit tensor. The material of the tube is considered to be incompressible, so the volume ratio J, $J=det(\mathbf{F})$, gives Eq. (2) expressing $J=1$.

$$\lambda_{zZ}\lambda_{rR}\lambda_{\theta\Theta}=1 \tag{2}$$

The equilibrium equations of a thin-walled tube with closed ends initially prestretched by axial force F_{red} and loaded by internal pressure P can be written in the form (3). Here σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} denote the radial, circumferential and axial component, respectively, of the Cauchy stress tensor $\boldsymbol{\sigma}$.

$$\sigma_{rr}=-\frac{P}{2},\quad \sigma_{\theta\theta}=\frac{Pr}{h},\quad \sigma_{zz}=\frac{Pr}{2h}+\frac{F_{red}}{2\pi rh} \tag{3}$$

The material of the tube is considered to be hyperelastic, described by the strain energy density function (elastic potential) W defined per unit reference volume. In this case, the constitutive equation relating components of the stress and strain tensor can be written in the form of (4). Here p denotes a Lagrangean multiplier reflecting the hydrostatic stress contribution (not captured in W, due to incompressibility) which has to be determined from the force boundary condition.

$$\sigma_{rr}=\lambda_{rR}\frac{\partial W}{\partial\lambda_{rR}}-p,\quad \sigma_{\theta\theta}=\lambda_{\theta\Theta}\frac{\partial W}{\partial\lambda_{\theta\Theta}}-p,\quad \sigma_{zz}=\lambda_{zZ}\frac{\partial W}{\partial\lambda_{zZ}}-p \tag{4}$$

Equations governing the inflation and extension of the thin-walled tube are obtained after substituting (4) into (3). The system