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Closed-form solutions for the contact problem of anisotropic materials indented by two collinear punches

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ABSTRACT

The present paper performs an exact analysis of the contact problem of anisotropic materials indented by two collinear punches. By considering the eigenvalue properties, the real fundamental solutions are given for the roots of the complex conjugate case and the real number case. A singular integral equation reduced from the stated problem is solved exactly for the case of two flat punches and two semi-cylindrical punches. Numerical results are presented to reveal how the interaction of the two punches and the elastic coefficient ratio affect the contact behaviors under the two collinear punches. When two individual rigid punches mingle as one rigid punch, classical results for one rigid punch are obtained, which validates the present derivation.

1. Introduction

Most mechanical systems consist of components that are in contact with each other. Being a principal way of transforming loads to a deformable body, contact is an important topic in solid mechanics [1]. The study of the contact problem finds its application in almost every corner of solid mechanics, from classical applications, such as material forming, drawing, molding, and machining to more ambitious applications such as projectile impacts, fluid–solid interactions, plate tectonics, and human joints.

Early in 1882, the first academic paper in this field, "On the contact of Elastic Solids," was published by Heinrich Hertz [2]. The Boussinesq–Flamant problem [3,4], which was defined as a half-space loaded by a point load, was the very first contact problem addressed by theoretical mechanics. Galin [5] developed the integral equation methods for the contact problem. Muskhelishvili [6] developed the methods of complex potentials and conformal maps and used them to address contact problems. Sneddon [7] used integral transforms, especially dual integral equations, for the actual solutions of the difficult boundary value problems of elasticity theory. His influence can be traced in much of the modern research on classical contact problems. Barber and Ciavarella [1] gave an outline of the development of the work on contact problems up to 2000. In addition to homogeneous materials, the contact problems of inhomogeneous materials have also been studied. Giannakopoulos and Suresh [8,9] solved the axis-symmetric contact problem of a graded half-space under a concentrated force or an axis-symmetric indenter, reporting that controlling the gradient variations of the substrate could change the contact tractions. Commercial finite element code [10] was also used to study the different indentation cases. When conventional ferrous materials are used in automotive brakes and clutches, frictional heating generation is a subject of concern to engineers. Recently, the thermo-mechanical contact problem of a finite graded layer fixed on a rigid foundation was studied, in which frictional heat is generated by a sliding cylindrical [11] or flat [12] punch. These contact problems are for isotropic or transversely isotropic materials.

Indentation measurements are often performed in order to handle anisotropic materials behavior requiring research on contact problems. Willis [13] investigated the anisotropic Hertzian contact problem through the use of double Fourier transforms. The punch and contact problems for an anisotropic elastic half-space were also studied by using Stroh's formalism [14–16]. Hwu and Fan [17] considered contact problems for two dissimilar anisotropic elastic bodies with relatively simple boundaries. Appling Stroh's formalism along with a conjugate gradient method-based (CGM-based) iteration scheme, Lin and Ovaert [18] established a model for solving the 2-D isothermal rough surface contact problem for general anisotropic materials. As analytical solutions were not widely available, numerical methods such as FEM and BEM can provide numerical values for the contact problems of anisotropic materials. Clements and Ang [19] utilized BEM to obtain a numerical solution for the generalized plane contact problem for inhomogeneous anisotropic elastic media, which

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involves the indentation of the elastic material by multiple straight rigid punches. An analytical solution for multiple rigid punches involving anisotropic materials needs to be presented to benefit the design and application of composite materials.

Motivated by the above-mentioned reasons, this paper establishes an exact analysis model of two collinear punches acting on the surface of the anisotropic materials. Real fundamental solutions are given in the case of the complex conjugate eigenvalue and the real eigenvalue. The originally considered problem is reduced to singular integral equations for the case of two flat punches and two semi-cylindrical punches. The singular integral equations are solved exactly. Explicit expressions of the stress components are given in terms of the elementary functions when two rigid punches mingle as one single punch [20]. Figures are drawn to show the influences of the interaction of two punches, such as the distance between the two rigid punches, and the elastic coefficient ratio on the contact behaviors subjected to the two collinear punches.

2. Problem statement and formulation

Consider semi-finite anisotropic materials, which are placed in an xoz Cartesian coordinates system. There are two punches on the $z=0$ plane symmetrically located with respect to $x=0$. That is, the punches are defined by $(z=0, a<x<b)$ and $(z=0, -b<x<-a)$ with each one pressed by the external loading P.

2.1. Basic equations

In a fixed rectangular coordinate system xoz, let u_k, σ_{pq}, and ε_{kl}, be the displacement, the stress, and the strain, respectively. The strain–displacement equations, the constitutive law, and the equations of the static equilibrium for linear, elastic anisotropic materials are given as follows:

$$\varepsilon_{kl}=\frac{1}{2}(u_{k,l}+u_{l,k}), \tag{1}$$

$$\sigma_{pq}=c_{pqkl}\varepsilon_{kl}, \tag{2}$$

$$\sigma_{pq,q}=0, \tag{3}$$

where repeated indices imply summation, a comma denotes differentiation, and c_{pqkl} are the elastic coefficients possessing full symmetry $c_{pqkl}=c_{qpkl}=c_{ipqlk}=c_{klpq}$. For y-independent problems in which all the quantities depend on x and z only, one can rewrite the generalized deformation field for a y-independent anisotropic problem as

$$u=u(x,z),\quad v=v(x,z),\quad w=w(x,z), \tag{4}$$

where u and w are the displacements in the xoz plane, and v is the anti-plane displacement perpendicular to the xoz plane.

2.2. Boundary conditions

Beneath each of the two punches, the surface contact stress is unknown and denoted as $p(x)$; outside the contact region, the contact stress is free. The surface shear stresses $\sigma_{xz}(x,0)$ and $\sigma_{yz}(x,0)$ are free on the surface. Thus, for each of the two punches, one has

$$\sigma_{zz}(x,0)=\begin{cases}-p(x), & a<|x|<b\\ 0, & |x|<a, |x|>b\end{cases}, \tag{5}$$

$$\sigma_{xz}(x,0)=0, \tag{6}$$

$$\sigma_{yz}(x,0)=0. \tag{7}$$

The equilibrium equation for each individual punch pressed by the external loading P should be satisfied such that

$$\int_{-b}^{-a}p(x)dx=\int_{a}^{b}p(x)dx=P. \tag{8}$$

At infinity, the following regularity conditions should be fulfilled:

$$u(x,z)\to 0,\quad \sqrt{x^2+z^2}\to\infty, \tag{9}$$

$$v(x,z)\to 0,\quad \sqrt{x^2+z^2}\to\infty, \tag{10}$$

$$w(x,z)\to 0,\quad \sqrt{x^2+z^2}\to\infty. \tag{11}$$

3. Real fundamental solutions

Substitution of Eq. (2) into Eq. (3) in view of Eq. (1) yields the governing equations,

$$c_{11}u_{,xx}+c_{55}u_{,zz}+c_{16}v_{,xx}+c_{45}v_{,zz}+(c_{13}+c_{55})w_{,xz}=0, \tag{12}$$

$$c_{16}u_{,xx}+c_{45}u_{,zz}+c_{66}v_{,xx}+c_{44}v_{,zz}+(c_{36}+c_{45})w_{,xz}=0, \tag{13}$$

$$(c_{13}+c_{55})u_{,xz}+(c_{36}+c_{45})v_{,xz}+c_{55}w_{,xx}+c_{33}w_{,zz}=0. \tag{14}$$

Here, the elastic coefficients c_{pqkl} are reduced to c_{pq}.

Using the Fourier sine (Eqs. (12) and (13)) or cosine (Eq. (14)) transform in terms of x and doing some manipulations, one can express the mechanical field in terms of the unknown functions $F_m(\eta)$ as follows:

$$\begin{Bmatrix}u(x,z)\\ v(x,z)\\ w(x,z)\end{Bmatrix}=\frac{2}{\pi}\int_0^\infty\sum_{m=1}^{3}\begin{Bmatrix}\Theta_{1m}(\eta,z)\sin(\eta x)\\ \Theta_{2m}(\eta,z)\sin(\eta x)\\ \Theta_{3m}(\eta,z)\cos(\eta x)\end{Bmatrix}F_m(\eta)d\eta, \tag{15}$$

where $\Theta_{km}(\eta,z)(k,m=1,2,3)$ are the components of the real fundamental solutions $\Theta_m=\left[\Theta_{1m}(\eta,z)\quad\Theta_{2m}(\eta,z)\quad\Theta_{3m}(\eta,z)\right]^T$ $(m=1,2,3)$ in the Fourier transformed domain with the superscript T denoting the transpose of a vector.

Here, the real fundamental solutions $\Theta_m=\left[\Theta_{1m}(\eta,z)\quad\Theta_{2m}(\eta,z)\right.$ $\left.\Theta_{3m}(\eta,z)\right]^T$ $(m=1,2,3)$ in the Fourier transformed domain are dependent on the following characteristic equations related to the governing equations Eqs. (12)–(14):

$$\begin{vmatrix}c_{11}-c_{55}\vartheta^2 & c_{16}-c_{45}\vartheta^2 & (c_{13}+c_{55})\vartheta\\ c_{16}-c_{45}\vartheta^2 & c_{66}-c_{44}\vartheta^2 & (c_{36}+c_{45})\vartheta\\ (c_{13}+c_{55})\vartheta & (c_{36}+c_{45})\vartheta & c_{33}\vartheta^2-c_{55}\end{vmatrix}=0. \tag{16}$$

The characteristic Eq. (16) can further be rewritten as $\beta_0\vartheta^6+\beta_1\vartheta^4+\beta_2\vartheta^2+\beta_3=0$ ($\beta_i(i=0,1,2,3)$ are dependent on material constants and omitted), which is a cubic equation in terms of ϑ^2. Thus, whether the root ϑ^2 is real or complex can be concluded from the parameter $\mathbb{Q}=(\ell_1/2)^2+(\ell_2/3)^3(\ell_1=-((\beta_1)^2/3(\beta_0)^2)+(\beta_2/\beta_0)$, $\ell_2=-(\beta_1\beta_2/3(\beta_0)^2)+(\beta_3/\beta_0)+((\beta_1)^3/27(\beta_0)^3))$ as follows:

A) $\mathbb{Q}>0$, one real root and one pair of conjugate roots;
B) $\mathbb{Q}=0$, three real roots: (I) $\ell_1=\ell_2=0$, $(\vartheta_1)^2=(\vartheta_2)^2=(\vartheta_3)^2=-(\beta_1/3\beta_0)$, and (II) $(\ell_1/2)^2=-(\ell_2/3)^3\neq 0$, $(\vartheta_1)^2\neq(\vartheta_2)^2=(\vartheta_3)^2$;
C) $\mathbb{Q}<0$, three real roots, $(\vartheta_1)^2\neq(\vartheta_2)^2\neq(\vartheta_3)^2$.

Generally speaking, only distinctive roots are available for anisotropic materials. Fundamental solutions $\Theta_m=\left[\Theta_{1m}(\eta,z)\quad\Theta_{2m}(\eta,z)\Theta_{3m}(\eta,z)\right]^T$ $(m=1,2,3)$ for real or complex conjugate roots can be obtained as follows:

Complex conjugate roots for ϑ

Given N_1 pairs of conjugate complex roots with positive real