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The principle of similitude analysed from plastic zones estimates ahead crack tips

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ABSTRACT

The similitude principle is the foundation for the use of Linear Elastic Fracture Mechanic (LEFM) because it permits to relate design nominal stress, σ_n, with experimental fracture toughness data. This relation is done assuming that if the laboratory specimen is under the same Stress Intensity Factor (K_I) of real structure, both will have the same fracture behaviour and both will have plastic zones (*pz*) equals from crack tip. However, this work shows that similar pieces with the same K_I have different *pz*, leading to the conclusion that the principle of similitude based on K_I is not completely accurate. This is also observed even under low load level where the disturbance generated in the stress field by the presence of the crack is located very close at its tip. The numerical *pz* estimates are done by Finite Element Method and two analytical stress fields are analysed, one is obtained from K_I and another obtained from K_I plus *T*-stress. In a different way that is advocated, this work shows that *T*-stress addition does not solve either *pz* estimates or the Principle of Similitude based on K_I.

1. Introduction

The principle of similitude was stated by Tolman [1] as follows "The fundamental entities out of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed exactly similar in every respect to the present universe." This principle has been applied in many fields of engineering. For example, it can be applied to fatigue crack growth in which the crack growth behaviour is reduced to material behaviour independent of crack length to describe physical phenomena [2]. Therefore, this principle allows relating the material response of simple laboratory specimens to real engineering structures [3]. It is also the basis of the Linear Elastic Fracture Mechanic (LEFM). However, it is necessary to consider a similitude parameter which relates the condition for similitude between the various geometries. In general, the parameter used to represent the similitude is the Stress Intensity Factor (SIF or *K*) in mode I, K_I. It is important to mention that this is just a simplification as the stress state in front a crack tip is non linear. The analytical stress field for a crack in an isotropic elastic solid material, independently of the geometry or remote load, could be represented by:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos(\theta/2) \begin{Bmatrix} 1-\ \sin(\theta/2)\sin(3\theta/2) \\ 1+\ \sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\sin(3\theta/2) \end{Bmatrix} + \begin{Bmatrix} Tstress \\ 0 \\ 0 \end{Bmatrix} \tag{1}$$

in which *r* and θ are polar coordinates and the *T*-stress is Williams' series constant or zero order term [4]. Structure components are designed to work in presence of cracks, thus, the validity of the parameter considered for the principle of similitude is or should be of considerable interest for engineers.

The ASTM [5] has recommended some requirements of a maximum allowable K_I, for example$K_I < (S_Y\sqrt{a})/\sqrt{2.5}$, where S_Y is the yield strength and *a* is the half-crack length. According to ASTM, under this condition one has a small-scale yielding (SSY), in which K_I is the toughness piece (K_{IC}) and Eq. (1) can be used [6]. It is guaranteed in SSY that the plastic zone (*pz*) size is much smaller than the crack length. It is important because the fatigue damage is depended of the elastic and plastic deformation histories of the piece [7,8].

The similitude principle is present in another important application of LEFM [9], e.g., Paris law [10]. This law is the basis for fatigue growth prediction and could be seen by:

$$\frac{da}{dN} = C(\Delta K)^n \tag{2}$$

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in which N is the number of load cycles and C and n are material properties. According to Paris law, for the same variation of K, the same crack growth rate will occur [9].

Due to the importance of the principle of similitude in the LEFM, this paper investigates the limitation of Eq. (1), which considers analytically the SIF and T-stress effects, to represent the linear elastic (LE) stress field around crack tips. It is analysed using an approach that uses pz estimates as reference. Furthermore, this paper shows that the parameter K_I used for similitude principle is not completely accurate, even for low nominal stress to yield strength ratio (σ_n/S_Y) where SSY is valid. For this purpose, the finite element method (FEM) is used to numerically obtain the complete LE stress field for plane stress and plane strain.

2. Background

Linear Elastic Fracture Mechanic (LEFM) was originally characterised by the Stress Intensity Factor (SIF), Eq. (1) with T-stress equal to zero. This task was made independently by Irwin [11] and Williams [12]. Both analysed Griffith's plate biaxially loaded in mode I. Irwin used Westergaard stress function and Williams expanded a stress function in terms of trigonometric series according to the boundary conditions of the problem. The stress field obtained from Westergaard stress function is given by the following equation:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} \mathrm{Re}(Z(z)) - y\mathrm{Im}(Z'(z)) \\ \mathrm{Re}(Z(z)) + y\mathrm{Im}(Z'(z)) \\ -y\mathrm{Re}(Z'(z)) \end{Bmatrix} \tag{3}$$

where Z is given by

$$Z(z) = \frac{z\sigma_n}{\sqrt{z^2 - a^2}} \tag{4}$$

in which $z=x+iy$ and $i=\sqrt{-1}$. The stress field obtained from Williams' series, written in polar coordinates is given by the following equation:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} \\ \frac{\partial^2 \Phi}{\partial r^2} \\ \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \Phi}{\partial \theta \partial r} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = r^{\lambda-1} \begin{Bmatrix} F''(\theta) + (\lambda+1)F(\theta) \\ \lambda(\lambda+1)F(\theta) \\ -\lambda F'(\theta) \end{Bmatrix} \tag{5}$$

in which Φ is Williams' series and F is a θ function that is related with Φ as $\Phi = r^{\lambda+1}F(\theta)$. Williams presented his series as:

$$\Phi = r^{\lambda+1}\Big\{ c_1 \sin[(\lambda+1)(\theta+\pi)] + c_2 \cos[(\lambda+1)(\theta+\pi)] + c_3 \sin[(\lambda-1)(\theta+\pi)] + c_4 \cos[(\lambda-1)(\theta+\pi)] \Big\} \tag{6}$$

The constants c_1, c_2, c_3 and c_4 are determined by assuming that the crack faces are free of load and by imposition to have finite strain energy at the crack tip. Free surface cracks implies $\sigma_{\theta\theta}(\pm\pi) = \sigma_{r\theta}(\pm\pi) = 0$, then $F(\pm\pi) = F(\pm\pi) = 0$ if $\lambda \neq 0$. So, $c_2 = -c_4$, $c_1 = (1-\lambda)/(1+\lambda) - c_3$. To have finite strain energy at the crack tip requires that $\sin(2\pi\lambda) = 0 \Rightarrow \lambda = n/2$, $n = 0, 1, 2, ..., \infty$. For mode-I, the resulting LE stress field given by Eq. (5) can be rewritten in Cartesian coordinates ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$) as is done as usual in the LEFM literature. Each i-th term of Williams' series can be written by θ-functions ($f_{ixx}, f_{iyy}, f_{ixy}$) according to the following equation:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos(\theta/2) \begin{Bmatrix} 1 - \sin(\theta/2)\sin(3\theta/2) \\ 1 + \sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\sin(3\theta/2) \end{Bmatrix} + \begin{Bmatrix} \eta \\ 0 \\ 0 \end{Bmatrix} r^0 + \kappa_2 r^{1/2} \begin{Bmatrix} f_{2xx}(\theta) \\ f_{2yy}(\theta) \\ f_{2xy}(\theta) \end{Bmatrix} + \kappa_3 r^{3/2} \begin{Bmatrix} f_{3xx}(\theta) \\ f_{3yy}(\theta) \\ f_{3xy}(\theta) \end{Bmatrix} + \ldots \tag{7}$$

in which η is the series constant term that was called of T-stress by Larsson and Carlsson [6]. Since then, LEFM has been characterised by two parameters. By analysing Eq. (7) it is possible to show that it is a generalisation of Eq. (1). Indeed, the use of T-stress to represent the stress field around crack tips in LE pieces was proposed by Irwin [13]. He did it to improve LEFM predictions of photo-elastic experimental results obtained by Wells and Post [14] for the stress field around crack tips. While investigating the limits recommended by ASTM for SIF use, Larsson and Carlsson [6] suggested the use of T-stress to adjust plastic zones (pz) estimated by LE analysis, approximating them to the pz obtained from elastic–plastic finite elements numerical analyses. Thereafter, the T-stress has been widely explored in the literature to model some interesting problems. In this context, it is worth mentioning the works of Rice [15], Leevers and Radon [16], Cardew et al. [17], Kfouri [18], Bilby et al. [19], Sham [20], Betegón and Hancock [21], Du and Hancock [22], O'Dowd and Shih [23], Nakamura and Parks [24], Wang and Parks [25], Wang [26], Hancock et al. [27], Kim et al. [28], Ganti and Parks [29], Zhang et al. [30], Ramesh et al. [31], Chen and Tian [32], Kang and Beom [33], Smith et al. [34], Zhao et al. [35], Chen et al. [36], Karihaloo and Xiao [37], Tan and Wang [38], and Su and Sun [39]. Fett [4] proved that T-stress is William's series constant or zero order term, and he showed how to evaluate the T-stress values for several geometries.

The most important application of LEFM, based in these two parameters, is the principle of similitude which affirms that pieces with the same K_I have similar fracture behaviour. This principle permits to relate the response of laboratory specimens with real structures. For example, the design nominal stress, σ_n, in the structures being considered is obtained from the laboratory fracture toughness data (K_I), Atkins and Caddell [40].

This work shows that the K_I parameter is not enough to represent the principle of similitude from the pz estimates. If K_I is the toughness of the laboratory specimen (K_{IC}), and Eq. (1) is used to represent the stress field for this specimen and for a real structure with different geometries and loads, but under the same K_I, both will have similar behaviour to fracture. This occurs because the pz formed at the crack tip of each piece are equals, i.e., both pieces have the same toughness. That is the reason for ASTM to define the SSY state that has a maximum allowable K_I and in which the size of pz is always smaller than the size of the crack. In the absence of these regulatory requirements there is a wide dispersion of results of toughness, which passes to depend of piece thickness and K_I can not be more associated to the piece toughness (K_{IC}). In this situation the parameter K_I used for the principle of similitude is not valid.

Using Eq. (1) to represent stress field under SSY conditions with Mises yield criterion, Unger [41] presented the following formulas for calculation of pz:

$$pz(\theta)_{pl-\sigma} = \left(K_I^2/2\pi S_Y^2\right) + \cos(\theta/2)^2\left[1 + 3\sin(\theta/2)^2\right] \tag{8}$$

$$pz(\theta)_{pl-\varepsilon} = \left(K_I^2/2\pi S_Y^2\right) + \cos(\theta/2)^2\left[(1-2\nu)^2 + 3\sin(\theta/2)^2\right] \tag{9}$$

in which ν is Poisson's coefficient. According to these classical estimates, the $pz(\theta)$ size directly ahead of crack tips, i.e. $\theta = 0^{\circ}$, in $pl-\sigma$, should be $pz(0)_{pl-\sigma} = pz_0 = (1/2\pi)(K_I/S_Y)^2$. This value is used as reference to normalise $pz(\theta)$ plots estimated in this work.

It is observed that most of studies have been carried out on determining analytical expressions for pz sizes for different types of materials. Xin et al. [42] studied pz size ahead of cracks tips in a plate made of orthotropic and isotropic materials. The expressions proposed by these works are generally based on SIF or obtained from numerical analyses. Besides, it is noticed that they compute pz estimates in parallel direction of crack plane, which is generally called by plastic radius (r_p) [43]. This is done because this plane is