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Path tracking and stability of a rolling controlled wheel on a horizontal plane by using the nonholonomic constraints

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ABSTRACT

This paper studies the stabilization, tracking of a predefined trajectory and how to reach a desired set point for a wheel which is rolling on a horizontal plane without slipping. For this purpose, the wheel is controlled by small torque generated by internal servomechanisms whose dynamics can be neglected. An efficient procedure to determine the kinetic energy of the wheel is developed by introducing a set of reference systems, which in combination with the Lagrange equations with multipliers allow deriving the mathematical model of the rolling wheel. In this model, the Euler angles, the coordinates of the plane–wheel contact point and a control law of proportional + integral + derivative (PID) type provide an efficient computational procedure to track arbitrary trajectories. It is shown that the nonholonomic constraints are fulfilled with admissible reaction forces, even when the desired trajectory has cusp points. A circumference and a family of astroids are used as trajectories to verify the motion conditions derived from the energy conservation and dynamical equilibrium of the wheel along such trajectories. The results of the analytical calculations are corroborated through numerical simulations.

1. Introduction

The nonholonomic mechanical problem of a wheel that rolls on a horizontal plane is emblematic in the history of rigid solid dynamics. Studies about this problem begun in the XIX century and were considered in the classical works of Chaplygin, Routh, Hamel, Appell and Korteweg, among others [1–4]. More recently, a wheel assimilated to a disk or a torus that rolls on a horizontal plane without involving control torques has been analyzed in Refs. [4–7]. On the other hand, the differential geometry approach currently offers a powerful tool in the new developments of mechanics such as nonholonomic systems [8–10] and the controlled motion of wheeled mobile robots, which have become the subject of numerous research studies [11–14] (see also references cited therein).

The stability of a wheeled vehicle is closely related to the kinematics and dynamics of a wheel, including friction and deformation [15,16] as well as effects of non-ideal contact between the surfaces of rolling bodies as it can be found in Refs. [4, Chapter IV; 17–19]. Currently, non-ideal contact problems of wheels rolling on surfaces of several materials can be found in Refs. [20,21] and in the references therein contained. On the other hand, the control and guidance of a rolling disk on a horizontal plane has been analyzed assuming that the motion of the disk is controlled by torques generated by internal servomechanisms, slender rods and rotors [22–24]. A prototype is the so called gyrover, which in essence is a single-wheel robot with a gyroscopic stabilization mechanism and an adequate pedaling torque [25].

In this work we derive the nonholonomic constraints of a rolling wheel on a horizontal plane through geometrical considerations and then we deduce the kinetic energy by using an adequate definition of a set of reference systems. Once the kinetic energy and the nonholonomic constraints have been obtained, the mathematical model of the wheel is deduced by using the Euler angles and the coordinates of the contact point between the wheel and the supporting plane. By using such mathematical model in combination with an adequate control law, we demonstrate the possibility of tracking of a predefined trajectory and reaching an arbitrary desired set point for the wheel. In a previous study [26], a rolling disk that can reach a predetermined set point (regardless the followed trajectory) has been investigated by using the Newton–Euler equations. However, the current paper aims to track of a predetermined trajectory for a rolling wheel that can be assimilated to a torus, and the nonholonomic constraints are different than the ones of Ref. [26]. Other examples of related rolling disks can be found in Refs. [17,22–24,27].

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To carry out our study, we first analyze the limit values of the rotation velocity to obtain a stable rolling wheel without control torques. Assuming that a friction rolling torque is present, the motion of the wheel is also analyzed, showing that the energy conservation principle is verified. Since the wheel without control torques is very unstable, a stabilizing torque around the leaning angle is added, showing that in this case the angular velocity of the wheel can be drastically reduced.

On the basis of the nonholonomic conditions it is shown that the wheel can be driven along a circumference. In this case, it is verified that the analytical expressions for the radius of the circle and the coordinates of the trajectory center are in agreement with numerical simulations. Besides, the conditions of dynamical equilibrium between the centrifugal force and the lateral reaction force at the contact point wheel–plane are fulfilled with a small stabilization torque [1–4,27].

The tracking of a prescribed trajectory is obtained by eliminating the Euler angles between the differential equations that define the coordinates of the contact point wheel–plane and the equations that arise from assuming fictitious forces applied at the mass center of the wheel. In the resulting equations, a control law of PID type is designed so that the error between the actual and the desired trajectory tends to zero. The stability conditions for the integral action of the PID controller are analyzed from Routh's criterion of stability [28]. This procedure has the advantage of obtaining small torques even when the control is applied abruptly, and thus the dynamics of the internal servomechanisms responsible for the control torques can be neglected. In addition it is shown that it is possible to track trajectories with cusp points and to jump to a prescribed set point from an arbitrary point of the tracked trajectory.

2. Nonholonomic constraints and mathematical model of the wheel

In this section, the nonholonomic constraints and the mathematical model of the wheel are analyzed. Let us consider a wheel modeled by a torus assuming that its mass is uniformly distributed along its surface and that its geometrical center coincides with the center of mass of the torus. In addition, the masses of the servomechanisms which will generate the control torques are assumed to be located in the mass center of the torus. As notation criterion, b shall denote the curvature radius of the torus meridian (i.e. the outer radius of the torus) and $a+b$ shall denote the radius of the equatorial circle of the torus. On the other hand, p, q and r are the precession, leaning and spin angles respectively, which define the orientation of the torus with respect to a fixed reference frame $OXYZ$. Thus $(\dot{p}, \dot{q}, \dot{r})$ are the corresponding angular velocities of the angles (p,q,r) defined with respect to the moving reference system $G\xi\eta\zeta$ bounded to the torus, as shown in Fig. 1a. The magnitudes (p,q,r) are the classical Euler angles (also denoted by $\psi \equiv p$, $\theta \equiv q$ and $\varphi \equiv r$ [1–4,6,27]), for which the considered notation in this paper aims to ease the discussion of the mathematical model. The magnitudes λ and μ are the reaction forces located in the OXY plane, which will be interpreted as Lagrange multipliers.

2.1. Nonholonomic constraints

Our first purpose is to determine the nonholonomic constraints of the torus when it is rolling on a horizontal plane, assuming that C is the only contact point between the torus and the supporting plane. To do this aim, from Fig. 1b it is deduced that the two curvature radii due to the rolling movement around point C are a and $CM = PG = b + a\ \sin\ q$. Assuming that dS_1 and dS_2 are infinitesimal displacements of the contact point C, we can write that:

$$dS_1 = adq; \quad dS_2 = (b + a\ \sin\ q)dr \tag{1}$$

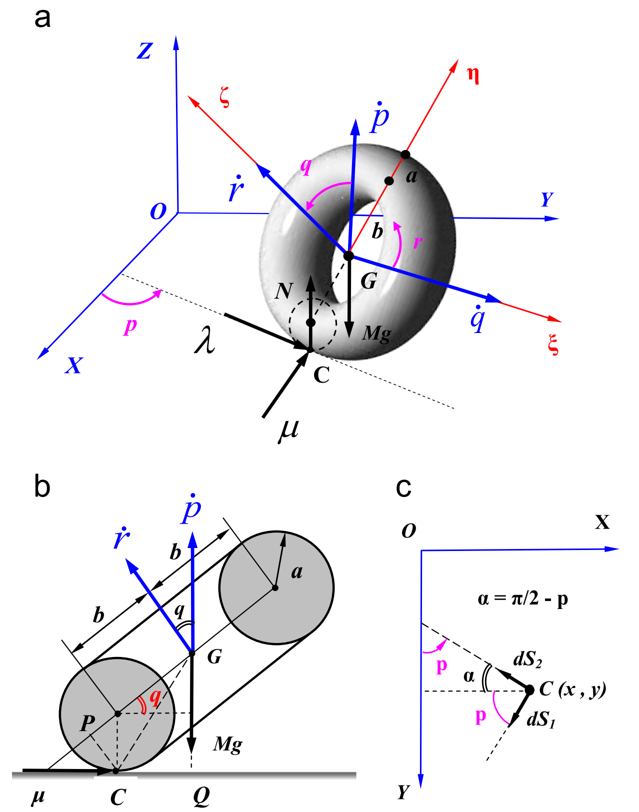

Fig. 1. (a) Wheel rolling on a horizontal plane with reaction forces λ and μ. The Euler angles and their derivatives are denoted by (p,q,r) and $(\dot{p}, \dot{q}, \dot{r})$ respectively. The inertial system is OXYZ whereas $G\xi\eta\varsigma$ is the reference system bound to the body. (b) Cross section of the wheel that is used to determine the generalized torques. (c) Scheme showing an infinitesimal displacement of the contact point C to determine the nonholonomic constraints. The parameter values are $a=0.1$ m, $b=0.3$ m, $M=5$ kg (wheel mass), $M_1=3$ kg (servomechanism mass) and $M_T=M+M_1=8$ kg. The equatorial and polar moments of inertia are $A_e=0.2875$ kg m^2 and $C_p=0.5250$ kg m^2 respectively.

where the curvature radius a is associated to the rotation defined by the leaning angle q and the curvature radius $b+a\ \sin\ q$ is due to the infinitesimal change of the spin angle r. The values of dS_1 and dS_2 are plotted in Fig. 1c. By projecting dS_1 and dS_2 on the OX and OY axes it follows that:

$$\left.\begin{aligned} dx &= dS_1\ \sin\ p - dS_2\ \cos\ p \\ dy &= -dS_1\ \cos\ p - dS_2\ \sin\ p \end{aligned}\right\} \tag{2}$$

On the other hand, substituting Eq. (1) into Eq. (2) and dividing the resulting equation by dt, Eq. (2) can be rewritten as

$$\left.\begin{aligned} \dot{x} &= -\dot{r}(b + a\ \sin\ q)\cos\ p + a\dot{q}\ \sin\ p \\ \dot{y} &= -\dot{r}(b + a\ \sin\ q)\sin\ p - a\dot{q}\ \cos\ p \end{aligned}\right\} \tag{3}$$

where the upper dot indicates derivative with respect to the time. Eq. (3) are the nonholonomic conditions assuming pure rolling of the torus on the horizontal plane. It should be noticed that the coordinates (x,y) of point C define the successive contact points between the torus and the supporting plane, which form a trajectory that is followed with a translation velocity $(\dot{x}, \dot{y})$. Such translation velocity is the one that an observer could perceive if he/she were unable of noticing the combined rotation motions of the wheel. However, the point of the torus in contact with the horizontal plane has zero relative velocity with respect to the $OXYZ$ reference system. In some Refs. [1–3,6] equivalent nonholonomic