International Journal of Mechanical Sciences 89 (2014) 482–486

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

Exact solutions for stresses, strains, and displacements of a rectangular plate with an arbitrarily located circular hole subjected to in-plane bending moment

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A R T I C L E I N F O

Article history:
Received 3 May 2014
Received in revised form
10 September 2014
Accepted 23 October 2014
Available online 31 October 2014

Keywords:
Perforated plate
Arbitrarily located circular hole
In-plane bending moment
Hoop stress
Airy stress function
Stress concentration factor

A B S T R A C T

Exact solutions for stresses, strains, and displacements of a rectangular plate with arbitrarily located circular hole subjected to in-plane bending moment are investigated by two-dimensional theory of elasticity using the Airy stress function. The present method of analysis is much simpler, but it produces an exact solution, which is its great strength, than the methods used by previous researchers. The hoop stresses occurring at the edge of the non-central circular hole are computed and plotted. The stress concentration factors (the maximum non-dimensional hoop stresses) depending on the location and size of the non-central circular hole are tabularized.

1. Introduction

Numerous researchers have investigated the mechanical behaviors of perforated plates, with main concerns being classified into three categories; stress concentration [1–29], vibration, buckling, and fatigue. The various discrete methods have been used to study them. The finite element method (FEM) is the most widely used for this perforated plate problems. Diverse methods other than FEM have been used like the complex varia.ble method, three-dimensional stress analysis, the Ritz method, the boundary element method, the differential quadrature element method, semi-analytical solution method, experimental method, conjugate load/displacement method, and Galerkin averaging method. Most of the shapes of perforated holes have three types of circular, elliptical, and rectangular cutout. Exact solutions for perforated plates with a non-central circular hole loaded by in-plane moment have not been reported.

In the present study, exact solutions for stresses, strains, and displacements of a perforated rectangular plate by a non-central circular hole subjected to in-plane bending moment are investigated by two-dimensional theory of elasticity using the Airy stress function. The hoop stresses occurring at the edge of the non-central circular hole are computed and plotted. The stress concentration factors (SCF) which is the maximum non-dimensional hoop stresses, depending on the location and size of the non-central circular hole are tabularized. Stress intensity factor (SIF) is often confused with SCF. The SIF is a scaling factor used in fracture mechanics to denote the stress intensity at the tip of a crack of known size and shape.

2. Airy stress function

Fig. 1 shows a perforated rectangular plate of lateral dimensions $L \times h$ by an arbitrarily located circular hole of radius of R under in-plane bending moment M_0 at $X = -L/2$ and $L/2$. The plate is assumed to be very large compared with the circular hole. The origin of the rectangular coordinate system (X, Y) is located at the center of the rectangular plate. The origins of the other rectangular coordinate system (x, y) and the polar coordinate one (r,θ) coincide with the center of the non-central circular hole. The axes of x and y are parallel with X and Y axes, respectively. The center of the non-central circular hole is located at $(X, Y) = (a, b)$.

First of all, considering a rectangular plate with no hole subjected to in-plane moment M_0, the stress components through the plate neglecting body forces are

$$\sigma_{XX}^0 = \frac{\partial^2 \phi^0}{\partial Y^2} = -\frac{M_0}{I} Y, \sigma_{XY}^0 = -\frac{\partial^2 \phi^0}{\partial X \partial Y} = 0, \sigma_{YY}^0 = \frac{\partial^2 \phi^0}{\partial X^2} = 0 \quad (1)$$

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http://dx.doi.org/10.1016/j.ijmecsci.2014.10.019
0020-7403/

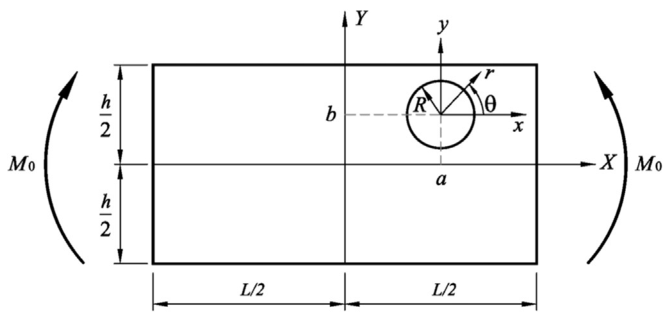

Fig. 1. A rectangular plate perforated by a non-central circular hole loaded by in-plane moment M_0.

where ϕ^0 is a fundamental Airy stress function; σ^0_{XX} and σ^0_{YY} are the normal stresses in X and Y directions, respectively, and σ^0_{XY} is the shear stress; and $I(=th^3/12)$ is the second moment of inertia of cross-sectional area of a rectangular plate with thickness of t. The fundamental Airy stress function ϕ^0 satisfies the governing equation $\nabla^4\phi^0=\nabla^2(\nabla^2\phi^0)=0$ with no body forces in 2-D plane problems in elasticity, where the Laplacian operator ∇^2 is expressed as

$$\nabla^2=\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2} \tag{2}$$

and ∇^4 is the bi-harmonic differential operator defined by $\nabla^2(\nabla^2)$ and becomes

$$\nabla^4=\nabla^2(\nabla^2)=\frac{\partial^4}{\partial X^4}+2\frac{\partial^4}{\partial X^2\partial Y^2}+\frac{\partial^4}{\partial Y^4} \tag{3}$$

in the rectangular coordinates (X,Y). From Eq. (1) the fundamental Airy function ϕ^0 can be assumed as

$$\phi^0=-\frac{M_0}{6I}Y^3+AY+B \tag{4}$$

where A and B are arbitrary integration constants. Since the relation of $Y=y+b$, Eq. (4) becomes

$$\phi^0=-\frac{M_0}{6I}(y+b)^3+A(y+b)+B \tag{5}$$

A linear function of x or y and a constant in the Airy stress function are trivial terms which do not give rise to any stresses and strains [30]. Dropping the trivial terms in Eq. (5), the fundamental Airy stress function ϕ^0 becomes

$$\phi^0=-\frac{M_0}{6I}(y^3+3by^2) \tag{6}$$

Using the relations of

$$y=r\sin\theta \tag{7}$$

and the multiple angles formulas

$$\sin^2\theta=\frac{1-\cos 2\theta}{2} \tag{8}$$

$$\sin^3\theta=\frac{3\sin\theta-\sin 3\theta}{4} \tag{9}$$

The fundamental Airy stress function ϕ^0 in Eq. (6) can be transformed into the bi-harmonic functions in the polar coordinates (r,θ) as below

$$\phi^0=-\frac{M_0}{24I}(3r^3\sin\theta-r^3\sin 3\theta-6br^2\cos 2\theta+6br^2) \tag{10}$$

which satisfies the governing equation $\nabla^4\phi^0=\nabla^2(\nabla^2\phi^0)=0$, where ∇^2 is

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{11}$$

and $\nabla^4=\nabla^2(\nabla^2)$ is expressed as

$$\nabla^4=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{12}$$

in the polar coordinates (r,θ). From the following relations between stresses and the Airy stress function, the stresses in the rectangular plate with no hole subjected to in-plane moment M_0 can be calculated in the polar coordinates as below

$$\sigma^0_{rr}=\frac{1}{r}\frac{\partial\phi^0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi^0}{\partial\theta^2}=-\frac{M_0}{4I}(r\sin 3\theta+r\sin\theta+2b\cos 2\theta+2b) \tag{13}$$

$$\sigma^0_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi^0}{\partial\theta}\right)=-\frac{M_0}{4I}(r\cos 3\theta-r\cos\theta-2b\sin 2\theta) \tag{14}$$

$$\sigma^0_{\theta\theta}=\frac{\partial^2\phi^0}{\partial r^2}=-\frac{M_0}{4I}(3r\sin\theta-r\sin 3\theta-2b\cos 2\theta+2b) \tag{15}$$

Let us return to the original problem of a rectangular plate with a non-central circular hole. The total Airy function ϕ becomes

$$\phi=\phi^0+\phi^* \tag{16}$$

where ϕ^* is an Airy stress function to cancel unwanted traction due to ϕ^0 on $r=R$. The normal and shear stresses on $r=R$ must be free as below

$$\sigma_{rr}|_{r=R}=[\sigma^0_{rr}+\sigma^*_{rr}]_{r=R}=0 \tag{17}$$

$$\sigma_{r\theta}|_{r=R}=[\sigma^0_{r\theta}+\sigma^*_{r\theta}]_{r=R}=0 \tag{18}$$

Therefore, σ^*_{rr} and $\sigma^*_{r\theta}$ on $r=R$ must have terms of $\sin\theta$, $\sin 3\theta$, $\cos 2\theta$, or a constant and have $\cos\theta$, $\cos 3\theta$, or $\sin 2\theta$, respectively, in order to eliminate the stresses σ^0_{rr} and $\sigma^0_{r\theta}$ on $r=R$ due to ϕ^0 in Eqs. (13) and (14). Tables 1 and 2 show the potential candidates of the bi-harmonic functions for the present problem from the tables by Dundurs [30], which contain stresses and displacements of certain bi-harmonic functions in the polar coordinates. However, the terms of $r^3\sin\theta$, $r^3\sin 3\theta$, $r^2\cos 2\theta$, and r^2 in the fundamental Airy stress function ϕ^0 of Eq. (10) must be excluded in ϕ^* in order not to disturb the traction in Eq. (1) at infinity. The terms of $r\ln r\sin\theta$, $r^2\ln r$, and $r\theta\cos\theta$ give rise to muti-valued displacements u_r and/or u_θ, in the directions of r and θ, respectively. Singularity at infinity occurs in stresses and displacements because of the terms of $r^4\cos 2\theta$ and $r^5\sin 3\theta$. Therefore, the total Airy stress function ϕ in Eq. (16) becomes

Table 1
Stresses of potential candidates of bi-harmonic functions ϕ.

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\ln r$	$1/r^2$	0	$-1/r^2$
$r^2\ln r$	$2\ln r+1$	0	$2\ln r+3$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^2\cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^4\cos 2\theta$	0	$6r^2\sin 2\theta$	$12r^2\cos 2\theta$
$\cos 2\theta/r^2$	$-6\cos 2\theta/r^4$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$
$\cos 2\theta$	$-4\cos 2\theta/r^2$	$-2\sin 2\theta/r^2$	0
$r^3\sin 3\theta$	$-6r\sin 3\theta$	$-6r\cos 3\theta$	$6r\sin 3\theta$
$r^5\sin 3\theta$	$-4r^3\sin 3\theta$	$-12r^3\cos 3\theta$	$20r^3\sin 3\theta$
$\sin 3\theta/r^3$	$-12\sin 3\theta/r^5$	$12\cos 3\theta/r^5$	$12\sin 3\theta/r^5$
$\sin 3\theta/r$	$-10\sin 3\theta/r^3$	$6\cos 3\theta/r^3$	$2\sin 3\theta/r^3$