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On the boundedness of motion in the spatial circular restricted three-body problem

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ABSTRACT

In this paper, we consider the existence conditions of bounded motions of an infinitesimally small particle in the spatial circular restricted three-body problem. In particular, by using the Jacobi integral that corresponds to the inertial coordinate system, we prove a theorem on boundedness of motion that enables us to supplement the Hill approach in the study of motion of an infinitesimally small particle.

1. Introduction

The circular restricted three-body problem (for mass points) is the three-body problem considered in the case where one of these bodies has a mass, which is so small that we can neglect the influence of this mass on circular orbits of two other bodies [1,2]. Nevertheless, until now, the three-body problem of this kind is still attractive for many investigators who can suggest a lot of interesting applications of this model [1,3–9].

As it was shown by Jacobi, the circular restricted three-body problem has the first integral. Due to this property, Hill proved [10] the existence of bounded motions for the small particle under the condition that the level constant h of the Jacobi integral is negative and $|h|$ exceeds a critical value $h^* > 0$. In what follows, we call it the Hill condition. Under this condition, the domain of possible motions of the infinitesimally small particle is a union of the domain ω_H (the Hill domain), which consists of motions bounded in coordinates, and the domain ω_{nc}, which consists of motions bounded in velocities, i.e., $\omega = \omega_H \bigcup \omega_{nc}$, and moreover, $\omega_H \bigcap \omega_{nc} = \varnothing$.

Possible examples of the domains ω_H and ω_{nc} are schematically shown in Figs. 1 and 2. Of course, their structure depends on $|h|$ (for details, see [1]). In these figures, the axis Oz is perpendicular to their plane. The shaded part corresponds to the "forbidden" region, where the motion of the small particle does not occur. We can see that, unlike ω_H, the domain ω_{nc} is not bounded and, in this domain, the problem of boundedness of the motion of the small particle arises. Nevertheless, in relation to the structure of ω_{nc} we have the following encouraging observation: the motions belonging to ω_{nc} satisfy the distality condition [11], which is extremely important in the framework of the proposed approach.

In what follows, the small particle motions belonging to the Hill domain ω_H are called Hill-stable motions. Among these motions, in addition to ones that are bounded in coordinates and velocity, there are motions that are bounded in coordinates, but not bounded in velocity, since these motions allow collisions between the small particle and massive bodies. The trajectories corresponding to motions with unbounded velocity will be referred to as special ones. In this case, as it is known, one of the mutual distances is zero.

Nowadays, among all possible variants of the restricted problem, a notable place is occupied by the Earth–Moon–Spacecraft system. Within its framework, the Moon was intensively studied at the end of the last century (USA, USSR). To save rocket propellant of space vehicles during flights to the Moon and other objects of the solar system, it is quite natural to use segments of special trajectories

In what follows, for the motions in the domain ω_{nc}, where we have no collisions and the distality condition is satisfied, it is important to determine whether the motion is bounded in coordinates. As it was shown by the author recently [12], if h is negative and the absolute value of h is sufficiently large and if, additionally, $\omega_{nc} \neq \emptyset$, i.e., under the Hill condition, the motions in the domain ω_{nc} are Lagrange stable under the additional condition that the circular problem is planar. Thus, in this case, the motions are bounded both in velocities and in coordinates. In the present paper, we succeeded to extend Theorem 1 from [12] to the spatial case. In this connection it should be pointed out that it is not possible to transfer automatically the proof of the boundedness of motions in the domain ω_{nc} in the planar case to the case of spatial motions. To this end, we had to find an additional resource. This resource was found in the structure of the Jacobi integral that corresponds to the inertial coordinate system.

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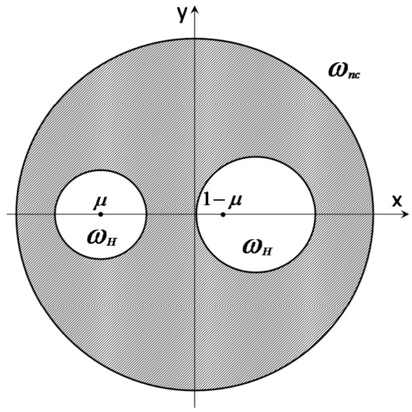

Fig. 1. This figure shows the domains ω_H and ω_{nc}.

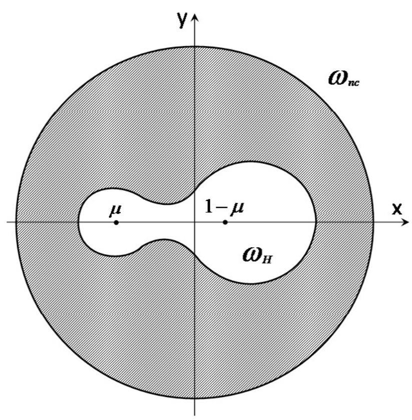

Fig. 2. This figure shows the deformation of ω_H when $|h|$ decreases.

So, we consider the circular restricted three-body problem in the case where the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, which are solutions of the two-body problem, correspond to circular orbits of points with masses m_1 and m_2. Passing over to relative vector lengths [13]:

$$\rho_i = \frac{\mathbf{r}_i}{|\mathbf{r}_{12}|}, \tag{1.1}$$

where $|\mathbf{r}_{12}| = |\mathbf{r}_{12}|_0 = \text{const}$, we write down the motion equations in the form

$$\begin{aligned} \rho_1'' &= \mu \frac{\rho_{12}}{|\rho_{12}|^3}, \\ \rho_2'' &= -(1-\mu)\frac{\rho_{12}}{|\rho_{12}|^3}, \\ \rho_3'' &= -(1-\mu)\frac{\rho_{13}}{|\rho_{13}|^3} - \mu\frac{\rho_{23}}{|\rho_{23}|^3}. \end{aligned} \tag{1.2}$$

Here, $\rho_{ij} = \rho_j - \rho_i$ $(i, j = 1, 2, 3)$, and the prime sign denotes the differentiation operation with respect to dimensionless time

$$\tau = \frac{\sqrt{G(m_1+m_2)}}{|\mathbf{r}_{12}|_0^{3/2}} t,$$

where $G > 0$ is the gravitation constant and

$$\mu = \frac{m_2}{m_1+m_2}, \quad 0 < \mu \leq \frac{1}{2}.$$

Also, system (1.2) can be represented in the form

$$\begin{aligned} \rho_{12}'' &= -\frac{\rho_{12}}{|\rho_{12}|^3}, \\ \rho_3'' &= -(1-\mu)\frac{\rho_{13}}{|\rho_{13}|^3} - \mu\frac{\rho_{23}}{|\rho_{23}|^3}. \end{aligned} \tag{1.3}$$

Along with Eqs. (1.2) and (1.3), we also use equations that correspond to a coordinate system rotating at a unit angular velocity about an axis perpendicular to the plane of rotation of two massive bodies. In this case, the second vector equation of system (1.3) takes the following form [1]:

$$\begin{aligned} x'' - 2y' &= x - (1-\mu)\frac{x-\mu}{\rho_{13}^3} - \mu\frac{x+1-\mu}{\rho_{23}^3}, \\ y'' + 2x' &= y - (1-\mu)\frac{y}{\rho_{13}^3} - \mu\frac{y}{\rho_{23}^3}, \\ z'' &= -(1-\mu)\frac{z}{\rho_{13}^3} - \mu\frac{z}{\rho_{23}^3}. \end{aligned} \tag{1.4}$$

Here, $\rho_{13} = |\rho_{13}|, \quad \rho_{23} = |\rho_{23}|,$

$$\rho_{13}^2 = (x-\mu)^2 + y^2 + z^2, \quad \rho_{23}^2 = (x+1-\mu)^2 + y^2 + z^2, \tag{1.5}$$

where (x, y, z) are coordinates of the small particle with respect to the rotating coordinate system. Let us denote $(x, y, z)^T = \mathbf{r}$, $(\tilde{x}, \tilde{y}, z)^T = \rho_3$, where $(\tilde{x}, \tilde{y}, z)$ are coordinates of the small particle with respect to the inertial coordinate system. Then we arrive at equality $\mathbf{r}^2 = \rho_3^2$.

Further, it is convenient to rewrite equalities (1.5) as follows:

$$\rho_{13}^2 = -2\mu x + \mu^2 + \mathbf{r}^2, \quad \rho_{23}^2 = 2(1-\mu)x + (1-\mu)^2 + \mathbf{r}^2, \tag{1.6}$$

and this implies

$$\mathbf{r}^2 = \rho_3^2 = -\mu(1-\mu) + (1-\mu)\rho_{13}^2 + \mu\rho_{23}^2. \tag{1.7}$$

In connection with (1.7), we also note that we have

$$\rho_3'^2 = -\mu(1-\mu) + (1-\mu)\rho_{13}'^2 + \mu\rho_{23}'^2. \tag{1.8}$$

The Jacobi integral of system (1.4) has the following form:

$$x'^2 + y'^2 + z'^2 - \left(x^2 + y^2\right) - \frac{2(1-\mu)}{|\rho_{13}|} - \frac{2\mu}{|\rho_{23}|} = 2h, \quad h = \text{const}. \tag{1.9}$$

2. A theorem on boundedness of motion

For our further study, we represent Jacobi integral (1.9) in the initial inertial coordinate system. According to our notation, it takes the following form [13]:

$$\rho_3'^2 - 2(\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}') - 2\left(\frac{1-\mu}{\rho_{13}} + \frac{\mu}{\rho_{23}}\right) = 2h, \tag{2.1}$$

and, after the change of variables

$$\tilde{x} = x\cos\tau - y\sin\tau,$$

$$\tilde{y} = x\sin\tau + y\cos\tau,$$

it turns into the equality

$$\rho_3'^2 - 2(xy' - yx') - 2\left(x^2 + y^2 + \frac{1-\mu}{\rho_{13}} + \frac{\mu}{\rho_{23}}\right) = 2h. \tag{2.2}$$

Also, it is possible to represent (2.1) in the form

$$\rho_3'^2 - 2(\rho_3 \times \rho_3')|_z - 2\left(\frac{1-\mu}{\rho_{13}} + \frac{\mu}{\rho_{23}}\right) = 2h, \tag{2.3}$$

emphasizing that the expression $(\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}')$ on the left-hand side of (2.1) is the projection of the angular momentum $(\rho_3 \times \rho_3')$ to the axis Oz of the inertial reference frame. In what follows, depending on the case considered, we can use the most convenient form of equalities (2.1)–(2.3).

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