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Closed-form numerical formulae for solutions of strongly nonlinear oscillators

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ABSTRACT

Based on Gauss–Kronrod quadrature rule, this paper provides closed-form numerical formulae of the period, periodic solution and Fourier expansion coefficients for a class of strongly nonlinear oscillators. Firstly, the period derived in the form of definite integral is addressed by a new equation constructed according to the fundamental theorem of calculus. Then, an approximate closed-form expression of the period can be established by employing only a low-order Gauss–Kronrod quadrature formula. Changing the lower limit of the integral, the closed-form expression becomes a numerical formula that can give the periodic solution of the system. After this, according to the partial integration rule, the calculation of the Fourier coefficients is derived in a very concise form. In general, the relative error of the approximate period can be reduced to 1e−6 only by employing a 31-point Kronrod rule. Error magnitude of the period indicates the maximum error level of the periodic solution and Fourier coefficients. In addition, the proposed formulae are stable convergent and the exact solutions being their convergence limits. Three very typical examples are given to illustrate the usefulness and effectiveness of the proposed technique.

1. Introduction

The strongly nonlinear oscillator models arise in a large number of actual physical systems, such as nonlinear roll of ships [1], nonlinear response of sensor diaphragm [2] and nonlinear vibrations of mechanical systems [3]. Therefore, the study of the method of dealing with nonlinear differential equations has a long history.

Numerical methods are effective methods to deal with nonlinear equations. Algorithm is the core of numerical analysis. For instance, numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral [4], which is used to calculate the period of the nonlinear oscillator in traditional applications. To obtain the time response of the nonlinear oscillator, people often use the Runge–Kutta (RK) method. The RK methods are a family of implicit and explicit iterative methods, which include the routine called the Euler Method [5]. The RK computation procedure is popular, but it cannot directly obtain the amplitude- and phase–frequency response curves of the forced and damped system without the aid of a discrete signal sampling and analysis program. In addition, if one need to obtain unstable solutions and bifurcation points, the numerical continuation software such as MATCONT is essential. For the free oscillation system, the Fourier expansion coefficients of periodic solution are useful for examining resonance phenomena under periodic or quasiperiodic forcing [6]. The traditional numerical methods usually use the discrete Fourier transform programs DFT or FFT after obtaining the discrete periodic solution. Therefore, it will be a complicated process to obtain a curve in which the Fourier coefficient continuously changes with the initial amplitude. In contrast, analytical methods have their unique advantages in nonlinear analysis. One of the advantages is that the given formulae are presented in a closed form. Nonetheless, analytical procedures are still complicated and require more extensive mathematical research.

In classical analytic methods, the perturbation theory is most famous and has been widely used [7,8]. Due to the dependence on small parameters, the perturbation method is usually applied only to weakly nonlinear problems. But there are still some improved methods that have been proposed so that the perturbation method can be extended to strong nonlinear systems [9–11]. However, these methods still have their own limitations. Therein, a relevant study was proposed by Amore and his collaborators [12,13], which used the principle of minimal sensitivity to address the period of strongly nonlinear oscillators. Recently, an important new method called multiple scales Lindstedt–Poincare was proposed to address both free undamped and forced damped cubic–quintic Duffing oscillator [14,15], which can provide acceptable solutions for the case of strong nonlinearities.

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Another most commonly used method is the harmonic balance (HB) method [16]. This method is not limited by small parameters, but due to the use of iterative method, it is difficult to construct higher-order analytical approximations. Various improved HB methods have been proposed by researchers [17–19]. One of the representative methods was proposed by Wu et al. [20]. Their technique incorporates salient features of Newton's method and the HB method, having excellent accuracy of both period and corresponding periodic solution for the entire range of oscillation amplitude. However, it is still necessary to obtain higher-order Fourier approximations by iterative method.

There is also a very important method named homotopy analysis method (HAM) [21,22]. The HAM employed the concept of the homotopy from topology and, by introducing a non-zero auxiliary parameter and employing a general zero-order deformation equation, generated a convergent series solution for nonlinear systems. In essence, HAM completely abandons the small parameters and is really applicable to strongly nonlinear problems. But in a sense, HAM is a semi-analytical technique based on computer and symbolic computing software, that is, if there is no computer and symbolic computing software, it is difficult to obtain a sufficient high-order approximate solution for strongly nonlinear problems [23].

In summary, the disadvantage of the numerical method is that the results are usually given directly as discrete values, and the disadvantages of the analytical method lie in its complicated process and mathematical restrictions, which makes it difficult to achieve higher accuracy. Different from the previous methods, this paper presents a new numerical technique which gives closed-form expressions of the period, periodic solution and Fourier expansion coefficients for a class of strongly nonlinear oscillators. This technique has the efficiency and high accuracy of numerical methods and has the characteristic of stable convergence with the exact solution as the limit. The closed-form formulae are given so that they can make the relevant parameters of the strongly nonlinear oscillator become known variables, which can be introduced into the forced and damped equation for derivation and operation. This provides a new possibility for nonlinear analysis, which is different from the traditional numerical methods.

This paper is organized as follows. In Section 2, we introduce the closed-form expressions of the period, periodic solution and coefficients of Fourier expansion. A detailed description for the accuracy, error tolerance and convergence of the closed-form formulae is given in Section 3. Three very typical examples are given in Section 4 to further demonstrate the efficiency and accuracy of the formulae. The conclusion is made in Section 5.

2. Mathematical modeling

2.1. Approximate closed-form expression of the period

Consider a one-dimensional oscillatory system governed by

$$\ddot{x} + f(x) = 0, \dot{x}(0) = 0, x(0) = A \tag{1}$$

where the nonlinear restoring force $f(x)$ is odd, i.e. $f(-x) = -f(x)$ and satisfies $xf(x) > 0$ for $x \in [-A, A]$, $x \neq 0$. A is the oscillation amplitude. Because $\ddot{x} = \dot{x}(d\dot{x}/dx)$, we have

$$\dot{x} = \frac{dx}{dt} = \pm\sqrt{2\int_x^A f(\xi)\,d\xi} = \pm\sqrt{2(F(A) - F(x))} \tag{2}$$

where $F(x)$ is the antiderivative of the integrable function $f(x)$ over the interval $[0, A]$. Then the period of vibration is obtained as

$$T = \int_0^T dt = 2\sqrt{2}\int_0^A \frac{dx}{\sqrt{F(A) - F(x)}} \tag{3}$$

Clearly, the integrand has a singularity at $x = A$, where its value tends to $+\infty$.

Generally, the integrand cannot be integrated in closed form. Therefore, numerical integration methods are necessary for the period calculation. In numerical analysis, Gaussian quadrature rule is preferred for the approximation of the definite integral of a singularity function, usually stated as a weighted sum of function values at specified points within the domain of integration, namely

$$\int_{-1}^{1} g(x)\,dx \approx \sum_{j=1}^{n} \omega_j g(\chi_j) \tag{4}$$

where ω_j, χ_j are the weights and nodes at which to evaluate the function $g(x)$. The evaluation points χ_j are the roots of a polynomial belonging to a class of orthogonal polynomials. In general, higher order estimates can be calculated by increasing the degree of the polynomial. However, for the integrand in Eq. (3), even if the number of Gauss nodes has greatly increased, the estimation error of Gaussian quadrature is still difficult to reduce to the ideal value. This is entirely because the value of the integrand is infinite at the singularity so that it cannot be well evaluated by even higher degree polynomials. Hence, in the conventional method, higher order quadrature rule together with interval subdivision and adaptive algorithm are necessary for approximating the exact result.

In order to facilitate analysis, we consider changing the integrand in Eq. (3) to a form that does not contain singularities, so that a high accuracy closed-form expression of the period can be constructed using only low-order Gaussian quadrature formulae.

Firstly, we construct a function that has the expression of

$$C(x) = C(A) - \frac{2(A - x)}{\sqrt{F(A) - F(x)}} \tag{5}$$

The both sides differentiation of Eq. (5) gives

$$C'(x) = \frac{1}{\sqrt{F(A) - F(x)}} + \frac{1}{\sqrt{F(A) - F(x)}}\left(1 - \frac{F'(x)(A - x)}{F(A) - F(x)}\right) \tag{6}$$

where ($'$) denotes differentiation with respect to x. Its definite integral over $[0, A]$ is

$$\begin{aligned}\int_0^A C'(x)\,dx = C(A) - C(0) = \int_0^A \frac{dx}{\sqrt{F(A) - F(x)}} + \\ \int_0^A \frac{1}{\sqrt{F(A) - F(x)}}\left(1 - \frac{F'(x)(A - x)}{F(A) - F(x)}\right) dx\end{aligned} \tag{7}$$

Therefore, the definite integral in Eq. (3) can be calculated as follows:

$$\begin{aligned}T = \frac{4\sqrt{2}A}{\sqrt{F(A) - F(0)}} - \\ 2\sqrt{2}\int_0^A \frac{1}{\sqrt{F(A) - F(x)}}\left(1 - \frac{F'(x)(A - x)}{F(A) - F(x)}\right) dx\end{aligned} \tag{8}$$

Due to the derivative of the function $F(x)$ with respect to the variable x can be defined as

$$F'(x) = \lim_{A - x \to 0} \frac{F(x + A - x) - F(x)}{(A - x)} \tag{9}$$

there is

$$F'(x)(A - x) = F(A) - F(x) - o(A - x) \tag{10}$$

where $o(A - x)$ is a higher-order infinitesimal. Therefore, as $x \to A$, the integrand in Eq. (8) satisfies

$$\frac{1}{\sqrt{F(A) - F(x)}}\left(1 - \frac{F'(x)(A - x)}{F(A) - F(x)}\right) = \frac{o(A - x)}{(F(A) - F(x))^{3/2}} \to 0 \tag{11}$$

The singularity is eliminated.

In this paper, we select Gauss–Kronrod quadrature rule to approximate the definite integral in Eq. (8). Before applying the quadrature rule, the integral over $[0, A]$ must be changed into the integral over [-1, 1]. Assuming that

$$X_j = \frac{A}{2}\chi_j + \frac{A}{2} \tag{12}$$