International Journal of Non-Linear Mechanics 102 (2018) 53–61

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

# Non-linear bright solitary SH waves in a hyperbolically heterogeneous layer

D. Demirkuş

*Beykent University, Avalon Campus, 34550 Buyukcekmece, Istanbul, Turkey*

ARTICLE INFO

*MSC:*
00A69
35-XX
35B20
35Q55

*Keywords:*
Non-linear SH waves
Heterogeneous layer
Bright solitary waves

ABSTRACT

This work investigates the propagation of non-linear shear horizontal (SH) waves in a layer of finite depth overlying a rigid substratum. We assume that the layer consists of heterogeneous, isotropic, and incompressible hyper-elastic materials. By using the method of multiple scales, we show that the self-modulation of non-linear SH waves is governed by the non-linear Schrödinger (NLS) equation. Using known properties of solutions of NLS equation, we find that bright solitary SH waves can exist depending on the non-linear constitution of the layer. Consequently, not only the effect of the heterogeneity but also the effect of the non-linearity on the deformation field is discussed for these waves.

## 1. Introduction

Elastic waves are not dispersive in an unbounded homogeneous medium, but they become dispersive under repeated processes occurring at the boundaries of wave guides [1,2]. Dispersive elastic waves have found many important applications in certain areas such as seismology, geophysics, nondestructive inspection of material surfaces, and electronic signal processing devices. More information about applications and for reviews, we refer to Ewing [2], Achenbach [3], Farnell [4], and Maugin [5].

The effect of constitutional non-linearity on the propagation characteristics of dispersive elastic waves has been studied by many investigators such as Teymur [6–8], Maugin and Hadouaj [9], Mayer [10], Fu [11], Porubov and Samsonov [12], Ferreira and Boulanger [13], Pucci and Saccomandi [14], Ahmetolan and Teymur [15,16], Destrade et al. [17], Teymur et al. [18], and Demirkus and Teymur [19]. In [19], the propagation of non-linear shear horizontal (SH) waves in a homogeneous, isotropic, and compressible hyper-elastic layer overlying a rigid substratum was investigated. Moreover, the propagation of linear Love waves in a heterogeneous media was discussed by Hudson [20] and Avtar [21].

The aim of this work is to study the propagation of non-linear SH waves in a heterogeneous, isotropic, and incompressible hyper-elastic layer overlying a rigid substratum. Heterogeneity is varied with the depth hyperbolically, and uniform in any direction parallel to the boundaries. We use the method of multiple scales and strike a balance between the non-linearity and dispersion in the asymptotic analysis, to derive a non-linear Schrödinger (NLS) equation describing a self-modulation of non-linear SH waves. After investigating the self-modulation, we discuss the effect of the non-linearity on the propagation characteristics of SH waves via NLS equation. We find that the existence of bright solitary SH waves depends on the non-linear constitution of the layer. As a result, not only the effect of the heterogeneity but also the effect of the non-linearity on the deformation field is considered for bright solitary SH waves.

## 2. Formulation of the problem

The spatial and material coordinates of a point referred to the same rectangular Cartesian system of axes are $(x_1, x_2, x_3)$ and $(X_1, X_2, X_3)$, respectively. We consider a layer of uniform thickness $h > 0$, lying on a rigid semi-infinite substratum. The layer is in the region between the planes $X_2 = 0$ and $X_2 = h$. In addition, a semi-infinite substratum occupies the region $X_2 < 0$. We consider waves of SH type, so displacements in the $X_1$- and $X_2$-directions are taken equal to zero. Moreover, the motion is assumed to be uniform in the $X_3$-direction. The displacement in the $X_3$-direction is zero at the rigid boundary $X_2 = 0$. Furthermore, the boundary $X_2 = h$ is assumed to be free of traction. Therefore, an SH wave described by

$$x_k = X_K \delta_{kK} + u_3(X_\Delta, t)\delta_{k3} \tag{1}$$

is assumed to propagate along $X_1$-axis where $u_3 = u_3(X_\Delta, t)$ is the displacement in the $X_3$-direction, $t$ is the time, and $\delta_{kK}$ is the usual Kronecker symbol. Latin and Greek indices have the respective ranges $(1, 2, 3)$ and $(1, 2)$, and also the summation convention on repeated indices is implied in (1) and in the sequel. Subscripts preceded by a

*E-mail address:* dilekdemirkus@beykent.edu.tr.

https://doi.org/10.1016/j.ijnonlinmec.2018.03.003
Received 7 October 2017; Received in revised form 10 March 2018; Accepted 12 March 2018
Available online 23 March 2018

comma indicate partial differentiation with respect to material or spatial coordinates.

In the absence of body forces, the equations of the motion in the reference state are

$$T_{\Delta\beta,\Delta} + T_{3\beta,3} = 0, \quad T_{\Delta3,\Delta} + T_{33,3} = \rho\ddot{u}_3 \tag{2}$$

where $T_{Kk}$ are the components of the first Piola–Kirchhoff stress tensor field accompanying the deformation field (1), a dot over $u_3$ indicates partial differentiation with respect to $t$, and $\rho = \rho(X_\Delta)$ is the density of the layer. The boundary conditions can be written as

$$T_{2k} = 0 \quad \text{on} \quad X_2 = h \quad \text{and} \quad u_3 = 0 \quad \text{on} \quad X_2 = 0. \tag{3}$$

The components of the deformation gradient tensors $x_{k,K}$ and $X_{K,k}$ are as given in the list below

$$x_{\alpha,\Delta} = \delta_{\alpha\Delta}, \quad x_{\alpha,3} = 0, \quad x_{3,\Delta} = u_{3,\Delta}, \quad x_{3,3} = 1, \tag{4}$$

$$X_{\Delta,\alpha} = \delta_{\Delta\alpha}, \quad X_{\Delta,3} = 0, \quad X_{3,\alpha} = -u_{3,\Delta}\delta_{\Delta\alpha}, \quad X_{3,3} = 1 \tag{5}$$

for the deformation field (1) which is isochoric, i.e. $j = \det x_{k,K} = 1$. In addition to using the components (4) and (5), if the relations $t_{kl} = j^{-1}x_{k,K}T_{Kl}$ and $T_{Kl} = jX_{K,k}t_{kl}$ are taken into consideration, then the components of the Cauchy stress tensor $t_{kl}$ and of the first Piola–Kirchhoff stress tensor $T_{Kl}$ can be written as

$$\begin{aligned} &t_{\alpha\beta} = \delta_{\alpha\Delta}T_{\Delta\beta}, \quad t_{\alpha3} = \delta_{\alpha\Delta}T_{\Delta3}, \\ &t_{3\beta} = u_{3,\Delta}T_{\Delta\beta} + T_{3\beta}, \quad t_{33} = u_{3,\Delta}T_{\Delta3} + T_{33}, \end{aligned} \tag{6}$$

$$\begin{aligned} &T_{\Delta\beta} = \delta_{\Delta\alpha}t_{\alpha\beta}, \quad T_{\Delta3} = \delta_{\Delta\alpha}t_{\alpha3}, \\ &T_{3\beta} = -u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha\beta} + t_{3\beta}, \quad T_{33} = -u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha3} + t_{33}, \end{aligned} \tag{7}$$

respectively. Therefore, the equations of the motion (2) in terms of $t_{kl}$ are expressed as follows:

$$\begin{aligned} &(\delta_{\Delta\alpha}t_{\alpha\beta})_{,\Delta} + (-u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha\beta} + t_{3\beta})_{,3} = 0, \\ &(\delta_{\Delta\alpha}t_{\alpha3})_{,\Delta} + (-u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha3} + t_{33})_{,3} = \rho\ddot{u}_3. \end{aligned} \tag{8}$$

If the layer consists of hyper-elastic materials, there exists a strain energy function $\Sigma$ which gives the mechanical properties of the constituent materials, and stress constitutive equations can be given by

$$T_{Kk} = \frac{\partial\Sigma}{\partial x_{k,K}}. \tag{9}$$

We consider that the constitutive materials are isotropic and heterogeneous, so $\Sigma$ is the function of the principal invariants of the Finger deformation tensor $\mathbf{c}^{-1}$ and $X_\Delta$, as

$$I = \mathrm{tr}\mathbf{c}^{-1}, \quad 2II = (\mathrm{tr}\mathbf{c}^{-1})^2 - \mathrm{tr}(\mathbf{c}^{-2}), \quad III = \det\mathbf{c}^{-1} \tag{10}$$

and calculated on the deformation field (1) as

$$I = II = 3 + K^2, \quad III = 1 \tag{11}$$

where $K^2 = u_{3,\Delta}u_{3,\Delta}$.

Let us now assume that the heterogeneity is varied only with the depth and uniform in any direction parallel to the boundaries and consider generalized neo-Hookean materials. Hence, the strain energy function $\Sigma$ has a form

$$\Sigma = \Sigma(I, X_2). \tag{12}$$

Then the stress constitutive equations are

$$t_{kl} = 2\frac{d\Sigma}{dI}(-\delta_{kl} + c^{-1}_{kl}) \tag{13}$$

where $c^{-1}_{kl} = x_{k,K}x_{l,K}$ are the components of Finger deformation tensor, found for the deformation field (1) as follows:

$$c^{-1}_{\alpha\beta} = \delta_{\alpha\Delta}\delta_{\beta\Delta}, \quad c^{-1}_{\alpha3} = \delta_{\alpha\Delta}u_{3,\Delta}, \quad c^{-1}_{3\beta} = u_{3,\Delta}\delta_{\beta\Delta}, \quad c^{-1}_{33} = 1 + K^2. \tag{14}$$

By substituting (14) into (13), the stress constitutive equations for (1) are found to be

$$t_{\alpha\beta} = 0, \quad t_{\alpha3} = 2\frac{d\Sigma}{dI}\delta_{\alpha\Delta}u_{3,\Delta}, \quad t_{3\beta} = 2\frac{d\Sigma}{dI}u_{3,\Delta}\delta_{\beta\Delta}, \quad t_{33} = 2\frac{d\Sigma}{dI}K^2. \tag{15}$$

We assume that $\Sigma$ is an analytic function of $I$ around 3; then its Taylor series can be written as

$$\Sigma(I, X_2) = (1/1!)\Sigma'(3, X_2)(I-3) + (1/2!)\Sigma''(3, X_2)(I-3)^2 + \cdots \tag{16}$$

where the prime means ordinary differentiation with respect to $I$. In addition, all coefficients are differentiable functions of $X_2$, and $\Sigma(3, X_2) = 0$. We thus have

$$\Sigma'(I, X_2) = \Sigma'(3, X_2) + \Sigma''(3, X_2)K^2 + \cdots \tag{17}$$

When we substitute (17) into (15), the stress constitutive equations can be expressed as follows:

$$\begin{aligned} &t_{\alpha\beta} = 0, \quad t_{33} = 2\Sigma'(3, X_2)K^2 + \mathcal{O}(K^4), \\ &t_{\alpha3} = 2\left[\Sigma'(3, X_2) + \Sigma''(3, X_2)K^2\right]\delta_{\alpha\Delta}u_{3,\Delta} + \mathcal{O}(K^4), \\ &t_{3\beta} = 2\left[\Sigma'(3, X_2) + \Sigma''(3, X_2)K^2\right]u_{3,\Delta}\delta_{\beta\Delta} + \mathcal{O}(K^4). \end{aligned} \tag{18}$$

Even though the materials are generalized neo-Hookean in this work, a similar job can be completed successfully for compressible or incompressible materials under certain restrictions for homogeneous case, as done in [6,7]. At the end of procedure for the anti-plane motion, the first two equations in (8) are satisfied identically, and the third equation becomes

$$\left\{2\left[\Sigma'(3, X_2) + \Sigma''(3, X_2)K^2\right]u_{3,\Delta} + \mathcal{O}(K^4)\right\}_{,\Delta} = \rho\ddot{u}_3 \tag{19}$$

where $\rho = \rho(X_2)$. Let $X = X_1$, $Y = X_2$, $Z = X_3$, and $u = u_3$ be. Since our aim is to deal with small but finite amplitude wave motions, proceeding with the approximate equations, rather than the exact ones, will be more convenient. Then the approximate governing equation and boundary conditions involving terms not higher than the third degree in the deformation gradients are written as

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - c_T^2\left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right) - \frac{1}{\rho}\frac{\partial(\rho c_T^2)}{\partial Y}\frac{\partial u}{\partial Y} \\ &= n_T\left[\frac{\partial}{\partial X}\left(\frac{\partial u}{\partial X}\mathcal{K}(u)\right) + \frac{\partial}{\partial Y}\left(\frac{\partial u}{\partial Y}\mathcal{K}(u)\right)\right] + \frac{\mathcal{K}(u)}{\rho}\frac{\partial(\rho n_T)}{\partial Y}\frac{\partial u}{\partial Y}, \end{aligned} \tag{20}$$

$$\frac{\partial u}{\partial Y} + \frac{n_T}{c_T^2}\mathcal{K}(u)\frac{\partial u}{\partial Y} = 0 \quad \text{on} \quad Y = h \quad \text{and} \quad u = 0 \quad \text{on} \quad Y = 0, \tag{21}$$

where the linear shear wave velocity $c_T = \sqrt{\mu/\rho}$, the non-linear material function $n_T$, and $\mathcal{K}(u)$ are defined by

$$c_T^2 = \frac{2}{\rho}\Sigma'(3, Y), \quad n_T = \frac{2}{\rho}\Sigma''(3, Y), \quad \text{and} \quad \mathcal{K}(u) = \left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial u}{\partial Y}\right)^2, \tag{22}$$

respectively. It can be observed that the functions $\mu$, $\rho$ and $n_T$ are not constants in this case, whereas these functions are constants in [19]. Furthermore, the constituent material of the layer softens in shear if $n_T < 0$ and hardens if $n_T > 0$. When $n_T = 0$ in Eq. (20), we recover the governing equation for the linear SH waves [2]. In our analysis, the non-linear material function $n_T$ is a differentiable function of $Y$ and the following choices on the functions $\mu$ and $\rho$;

$$\mu = \mu_0\cosh^2(\alpha Y), \quad \rho = \rho_0\cosh^2(\alpha Y) \tag{23}$$

are made. Here, $\mu_0$ and $\rho_0$ are constants, and $\alpha$ is a parameter. The choices (23) are also used in [21]. The following sections will be based upon all discussions and assumptions above.

## 3. Self-modulation of non-linear SH waves

Now we investigate the self-modulation of non-linear SH waves with small but finite amplitude in a layer overlying a rigid substratum