International Journal of Non-Linear Mechanics 101 (2018) 68–76

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Magnetosonic waves traveling against a plasma wind

Manuel Núñez

Depto. Análisis Matemático, Universidad de Valladolid, 47011 Valladolid, Spain

A R T I C L E I N F O

MSC:
41A60
76L05
76W05
85A30

Keywords:
MHD critical points
Wavefront evolution
Geometric optics
Shock formation

A B S T R A C T

The points of a plasma flow where the velocity matches the wave velocity of either the slow or fast magnetohydrodynamics modes are known to be highly relevant e.g. in the study of stellar winds. We develop a geometric optics analysis of a simple flow geometry where slow or fast critical points are present. A first integral of the ray equations is useful for a qualitative study of the ray topology and the transport of wavefronts, which are illustrated with numerical examples. The evolution of perturbations traveling along rays and eventually becoming shocks is also analyzed, showing that whenever wavefronts are slowed either by a small wave velocity or the geometry of the ray, shocks are more likely to occur.

1. Introduction

Assume an ideal plasma in motion and start a magnetosonic wave at some point of it. If it moves along with the plasma flow, it is accelerated by it; if it goes upstream, it will move backwards or forward depending on the different velocities of the wave and the fluid; if originally the wavefront is oblique to the flow direction, a range of possible evolutions lies ahead. The problem seems technically interesting, but the fact that there exists a precise physical setting where this situation occurs makes it doubly relevant. This concerns the solar (and stellar) winds. The groundbreaking model of Parker [1,2] showed how the particles of the solar wind may reach supersonic velocities; later this was generalized to include the magnetic field by Weber and Davis [3], to non-radial flows [4], to relativistic settings [5], and much else. Today the subject may be considered classical [6,7], but much remains to be learned. In particular magnetosonic shocks are not covered by the original theory, which focuses on the properties and location of slow, Alfvén and fast critical points, i.e. those where the MHD wave velocity matches the one of the flow. The latter evolution of the waves, and eventual creation of shocks, is obviously important. In particular the bow shock of the Earth and other astrophysical objects is a fast magnetosonic one [8,9]; slow shocks are found in Earth's magnetosphere [10] and coronal plumes [11], fast ones in the heliosheath [12] and in solar flares [13]. These last papers are largely observational or use ad hoc numerical simulations, appropriate for the complexity of the physical setting, where an entirely analytical development is impossible. We intend to study how magnetosonic waves evolve in the vicinity of MHD critical points starting from first principles, so that a simple geometry is needed for the mathematics to be tractable. The MHD ideal equations are still nonlinear, but the nonlinear terms involve at most the product of two of the main variables (velocity, magnetic field and density) or their derivatives. This is the ideal scenario for the use of nonlinear geometric optics when studying high-frequency perturbations: [14] is a good exposition. One-phase expansions such as ours are studied in [15,16]. The subject has grown to become highly elaborate, but the review paper [17] remains an excellent account of the motivation and problems in geometric optics. The techniques of weakly nonlinear geometric optics have been used with other names such as theory of weak waves or rays in hyperbolic differential equations, with all their attendant applications to Fluid Dynamics and Magnetohydrodynamics: [18,19] are among the best monographs dealing with the subject. This paper is essentially divided into two parts: the first one deals with the shape of evolution of rays and wavefronts of fast and slow waves; and the second one considers the formation of shocks by a fast magnetosonic perturbation transported along rays. We omit Alfvén waves because they are much simpler for our geometry and because they do not give rise to shocks. This last question depends on the value of a certain integral whose integrand is a rather complex function of the equilibrium parameters, which must be estimated to indicate which rays are more likely to yield a shock. Where analytic solutions are impossible to find, we have taken typical values of the parameters and perform numerical integration to illustrate the general behavior and reach qualitative conclusions.

E-mail address: mnjmhd@am.uva.es.

https://doi.org/10.1016/j.ijnonlinmec.2018.02.003
Received 28 November 2017; Accepted 12 February 2018
Available online 16 February 2018

2. Equilibrium, rays and wavefronts

The correct geometry to deal with the solar wind would be a cylindrical or spherical one. however, since we wish to study the neighborhoods of the critical fast and slow points, we may take there the radius as roughly constant and use instead cartesian coordinates. We also assume that the flow follows the radii, as it happens with slowly rotating stars; and finally, we restrict our study to the equatorial plane, as in [3], i.e. to a two dimensional configuration. In fact, the magnetic field often has a vertical component, and the equilibrium equation is a Grad–Shafranov one; see e.g. [20]. Axisymmetry means now dependence only on one variable, which we set as x. The equilibrium equations in this situation are as follows. Denoting $\mathbf{v} = (v, 0)$ the fluid velocity, $\mathbf{B} = (B_1, B_2)$ the magnetic field, ρ the density, P the pressure, $\mathbf{F} = (F, 0)$ a forcing (e.g. gravitational), $' = d/dx$, the ideal MHD equations become

$$\rho v v' = -P' - B_2 B_2' + F,$$

$$B_1 B_2' = 0 \quad \text{(momentum equation)}, \tag{1}$$

$$(vB_2)' = 0 \quad \text{(induction equation)}, \tag{2}$$

$$B_1' = 0 \quad \text{(divergence equation of the magnetic field)}, \tag{3}$$

$$(\rho v)' = 0 \quad \text{(continuity equation)}. \tag{4}$$

Assuming $B_1 \neq 0$, (1) yields $B_2 = \text{const.}$, (2) implies $B_2 = 0$, so that $B_1 = B$ (const.) by (4). Hence we are left with

$$\rho v v' = -P' + F, \tag{5}$$

$$\rho v = C \quad \text{(const.)} \tag{6}$$

We will take units so that $C = 1$, and assume a polytropic plasma $P = A\rho^\gamma$. Traditionally one takes the monatomic constant $\gamma = 5/3$, and Parker took $\gamma = 1$ (isothermality). However, none of these values is entirely admissible as they yield $v(\infty) = \infty$ (for $\gamma = 1$) or $v(\infty) = 0$ (for $\gamma = 5/3$). It is found that $\gamma \sim 1.1 - 1.4$ [7]. We have taken $\gamma = 1.1$ for our modelization of the rays, and $\gamma = 1$ in the simplified form of the slow rays (since anyway we do not pretend to go far away from the slow critical point).

We know that high-frequency perturbations of this equilibrium are adequately described by the geometric optics model. We assume the basic facts known and start from the formula for the fast and slow magnetosonic velocities, $c_\pm$:

$$(c_\pm - \mathbf{v}\cdot\mathbf{n})^2 = \frac{1}{2}\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right) \pm \frac{1}{2}\left[\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right)^2 - 4\frac{1}{\rho}\frac{\partial P}{\partial \rho}(\mathbf{B}\cdot\mathbf{n})^2\right]^{1/2}, \tag{7}$$

where $\mathbf{n}$ represents the wave vector (normal to the wavefront), which is an eigenvector of the eikonal equation and $c_\pm$ its eigenvalue. In terms of the sound speed $c_s^2 = \partial P/\partial\rho$ and the Alfvén velocity $c_A^2 = B^2/\rho$, and writing for a two dimensional geometry $\mathbf{n} = (\cos\psi, \sin\psi)$, since $\mathbf{v} = (v, 0)$,

$$(c_\pm - v\cos\psi)^2 = \frac{1}{2}(c_s^2 + c_A^2) \pm \frac{1}{2}[(c_s^2 + c_A^2)^2 - 4c_s^2 c_A^2 \cos^2\psi]^{1/2}. \tag{8}$$

Calling

$$\mu = \frac{A\gamma}{B^2} = \frac{c_s^2}{c_A^2}\rho^{-\gamma} = \frac{c_s^2}{c_A^2}v^\gamma, \tag{9}$$

(notice that if $\gamma = 1$, then $\mu = c_s^2/B^2$), we have

$$c_\pm = v\cos\psi + Bv^{1/2}\left(\frac{1}{2}(1 + \mu v^{-\gamma}) \pm \frac{1}{2}\left[(1 + \mu v^{-\gamma})^2 - 4\mu v^{-\gamma}\cos^2\psi\right]^{1/2}\right)^{1/2}. \tag{10}$$

From now on we will denote either the fast or the slow speed by c, and by U the 'intrinsic' magnetosonic speed, i.e. the term to the right of $v\cos\psi$ in (10), not linked to the fluid velocity except for the density; the context will make clear which one are we referring to. Notice that always $U > 0$. Rays are bicharacteristic curves $(\mathbf{x}(t), \mathbf{n}(t))$ of the eikonal equation. They satisfy [19] the equations

$$\frac{d\mathbf{x}}{dt} = c\mathbf{n} + \frac{\partial c}{\partial \mathbf{n}} - \mathbf{n}\left(\mathbf{n}\cdot\frac{\partial c}{\partial \mathbf{n}}\right)$$

$$\frac{d\mathbf{n}}{dt} = \mathbf{n}\left(\mathbf{n}\cdot\frac{\partial c}{\partial \mathbf{x}}\right) - \frac{\partial c}{\partial \mathbf{x}}. \tag{11}$$

Once found the phase ϕ through the eikonal equation, wavefronts are surfaces (or lines) $\phi = \text{const}$. Since along the rays $d\phi/dt = 0$, wavefronts are transported by rays. In our case system (11) simplifies to

$$\frac{dx}{dt} = c\cos\psi - \frac{\partial c}{\partial \psi}\sin\psi \tag{12}$$

$$\frac{dy}{dt} = c\sin\psi + \frac{\partial c}{\partial \psi}\cos\psi \tag{13}$$

$$\frac{d\psi}{dt} = \frac{\partial c}{\partial x}\sin\psi. \tag{14}$$

Notice that since the only variables involved are (x, ψ), Eq. (13) uncouples from the rest and may be ignored, except to find $y(t)$. We have

$$\frac{d}{dt}\left(\frac{c}{\sin\psi}\right) = 0, \tag{15}$$

so that $c/\sin\psi$ is a first integral of the system. Thus the projection of a ray $t \to (x(t), y(t), \psi(t))$ into the (x, ψ) plane lies in one of the curves

$$\frac{c}{\sin\psi} = k \quad \text{(const.)} \tag{16}$$

Since c is even with respect to ψ and $\sin\psi$ is odd, these curves are symmetric with respect to $\psi = \pi$; the reflection with respect to $\psi = \pi$ of the curve associated to the constant k is the one associated to $-k$. Thus it is enough to study the geometry of the level curves of the function $c/\sin\psi$ when $\psi \in [0, \pi]$, $v \geq 0$. The lines $\psi = 0, \pi, 2\pi$ are singular of the first integral, but this is an artifact; they themselves are perfectly good rays, since ψ is constant if $\sin\psi(0) = 0$. These curves have a branching point at $c = 0$, $\sin\psi = 0$. Since $c = v\cos\psi + U$, this needs $\cos\psi < 0$, i.e. $\psi = \pi$. Let us find explicitly the value of v in the branching point by taking in (10) $\psi = \pi$.

For the slow wave,

$$v_- = Bv_-^{1/2}\inf\{1, \mu v_-^{-\gamma}\}, \tag{17}$$

and for the fast one,

$$v_+ = Bv_+^{1/2}\sup\{1, \mu v_+^{-\gamma}\}. \tag{18}$$

Therefore the next alternatives may occur:

If $\mu v_-^{-\gamma} \leq 1$, which implies $\mu v_+^{-\gamma} \leq 1$, $v_+ = B^2$, $v_- = (\mu B^2)^{1/(1+\gamma)}$. This occurs if $\mu \leq B^{2\gamma}$.

If $\mu v_+^{-\gamma} \geq 1$, which implies $\mu v_-^{-\gamma} \geq 1$, $v_- = B^2$, $v_+ = (\mu B^2)^{1/(1+\gamma)}$. This occurs if $\mu \geq B^{2\gamma}$.

The remaining alternative $\mu v_+^{-\gamma} < 1 < \mu v_-^{-\gamma}$ would imply $v_+ = v_- = B^2$, so it is contradictory. For low beta plasmas, the most likely alternative is the first one, since $\mu v_-^{-\gamma} \leq 1$ is equivalent to $c_s v^\gamma \leq c_A^{1+\gamma}$ at the point $v = v_-$. We now proceed to analyze in detail slow and fast rays and wavefronts.

2.1. Slow magnetosonic rays

For low beta plasmas $c_s \ll c_A$ we may ignore the terms c_s/c_A in (8), and we are left with

$$c \sim v\cos\psi + B\sqrt{\mu}v^{(1-\gamma)/2}|\cos\psi| = v\cos\psi + c_s|\cos\psi|. \tag{19}$$

Notice that in the isothermal case c_s is constant. As explained, this cannot hold all over the range of x, but it is a very good approximation for points not too far from the critical one. This approximation cannot hold either when $v \to 0$, because there the term $v^{-\gamma} \to \infty$, but anyway $v = 0$ is unphysical e.g. in the solar wind. To illustrate how c and its approximation compare, we have plotted some level curves of both for $B = 2$, $\mu = 0.5$, and $\gamma = 1.1$ for the exact velocity.