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Long wave instability of thin film flowing down an inclined plane with linear variation of thermophysical properties for very small Biot number

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ABSTRACT

We investigated interfacial instability of a thin liquid film flowing down an inclined plane, considering the linear variation of fluid properties such as density, dynamical viscosity, surface tension and thermal diffusivity, for the small variation of temperature. Using long wave expansion method and considering order analysis specially for very small Biot number (Bi) we obtained a single surface equation in terms of the free surface $h(x,t)$. Considering sinusoidal perturbation method we carried out linear stability analysis and obtained the critical Reynolds number (Re_c) and linear phase speed (c_r), both of which depend on K_μ, K_ρ but independent of K_σ, K_κ. Using the method of multiple scales, weakly nonlinear stability analysis is carried out. We demarcated subcritical, supercritical, unconditional and explosive zones and their variations for the variation of K_μ, K_ρ and K_σ. Also we discussed the variations of threshold amplitude in the subcritical as well as in the supercritical zones for the variation of K_μ, K_ρ and K_σ. Finally we discussed the variation of nonlinear wave speed N_{cr} for the variation of K_μ, K_ρ and K_σ.

1. Introduction

Searching the study of instability mechanism on the thin film we found that most of the authors studied the effect of isothermal/nonisothermal cases and very few of them considered the variation of different physical properties due to variation of temperature. Although certain physical properties such as viscosity, density, thermal conductivity and surface tension highly depend on the variation of temperature, so these physical quantities could not be considered as constant in real situation. Goussis and Kelly [1] studied the effects of variable viscosity on the surface wave mode of instability of a liquid film flowing down heated or cooled inclined surface by means of long wavelength analysis. They found that cooling stabilizes the flow, while heating destabilizes it. For the case of cooling they derived a cutoff Prandtl number above which the flow is always stable. The effects of variable viscosity on the surface wave mode of instability of a liquid film flowing down heated or cooled inclined surfaces, studied again by Goussis and Kelly [2], using two models of viscosity variation with temperature. Both the models confirmed that heating destabilizes the flow, while cooling stabilizes it. They also found that the critical wave number is always zero in case of heating, while for the case of cooling critical wave number can be non zero. Reisfeld and Bankoff [3] studied the stability of a heated volatile liquid film subjected to surface tension and Vander Waals forces, assuming the linear variation of viscosity with temperature. They found that effect of variable viscosity reduce the rupture time of the film relative to the constant viscosity. Pascal et al. [4] first studied the long-wave instability of flow down an inclined plane considering the linear variation of different thermophysical properties such as density, conductivity, surface tension, viscosity and specific heat. Linear stability analysis is carried out by long wave perturbation method. They found the critical Reynolds number considering the cases when Biot number $Bi = 0$ or $Bi \longrightarrow \infty$ or as $\Lambda = \lambda = 0$, where Λ and λ are the scaled gradients with respect to temperature of the thermal conductivity and viscosity respectively. For the general case they obtained an analytical expression of the critical Reynold's number by implementing asymptotic expansions as $Bi = 0$ or as $\Lambda = \lambda = 0$. Recently D'Alessio et al. [5] studied the effects of variable fluid properties on thin film instability, where the fluid properties are allowed to vary linearly with temperature except the specific heat C_p which is taken as constant. Linear stability analysis is carried out using the long wave perturbation method. They showed how the critical Reynolds number and perturbation phase speed depend on the various dimensionless parameters and thermophysical properties. The approach developed by Pascal et al. [4] and S.J.D. D'Alessio et al. [5] although are pioneering work, restrict themselves to discuss only linear stability analysis, due to lengthy process of algebraic computation. But our proposed analysis helps to construct a single surface equation in terms of free surface

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$z = h(x, t)$, which helps us to discuss the linear as well as weakly nonlinear stability analysis. For simplicity we have restricted ourselves only for small Biot number. Our proposed analysis is purely analytical and not very lengthy, so that we may easily avoid the computation of the problem arises due to variations of several thermophysical properties.

Biot number Bi is the ratio of the heat transfer resistance inside the thin film and the free surface in the contact of ambient air. When a constant thermal gradient is applied to the inclined plane which is greater than the temperature of ambient air, the very small Biot number physically interprets that the heat conduction inside the liquid film is much faster than the heat convection away from its free surface.

In this article, we studied, the instabilities of the gravity driven flow of thin liquid film by long wave expansion method and carried out linear as well as weakly nonlinear stability analysis considering variable thermophysical properties such as density, dynamical viscosity, surface tension and thermal diffusivity for very small Biot number.

2. Formulation of the problem

Let us consider a two dimensional flow of viscous film flowing along an inclined plane of inclination $\gamma (0 \leq \gamma \leq \frac{\pi}{2})$ with the horizon. The origin is fixed on the inclined plane, the x-axis is chosen along the inclined plane in the downhill direction and the z-axis pointing in the upward normal direction. We have neglected the effect of latent heat due to evaporation by assuming the liquid to be nonvolatile. Also, it is assumed that the plane along which the thin film is flowing is a perfect heat conductor being heated uniformly from below with temperature T_w, which is higher than the ambient air temperature T_0 above the film. We have also ignored the dynamic influence of the ambient air. The temperature difference $\triangle T = T_w - T_0$ is responsible for heating the thin film. It is a fact that some fluid properties such as density ρ, dynamic viscosity μ, thermal diffusivity κ and surface tension σ are temperature dependent. Variations of these properties cannot be totally ignored with the variations of temperature T, though consideration of these variations makes the problem very difficult to handle. However, there are many situations of practical occurrence in which the basic equations can be simplified considerably by taking appropriate approximations, specially when their variations in the temperature is moderate amounts only.

It is assumed that, for moderate temperature variation, the density, dynamical viscosity, surface tension and thermal diffusivity all varies linearly with temperature as:

$$\rho(T) = \rho_0[1 - K_\rho(\frac{T - T_0}{\Delta T})] \tag{1}$$

$$\mu(T) = \mu_0[1 - K_\mu(\frac{T - T_0}{\Delta T})] \tag{2}$$

$$\sigma(T) = \sigma_0[1 - K_\sigma(\frac{T - T_0}{\Delta T})] \tag{3}$$

$$\kappa(T) = \kappa_0[1 - K_\kappa(\frac{T - T_0}{\Delta T})] \tag{4}$$

where ρ_0, μ_0, σ_0 and κ_0 are the values of ρ, μ, σ and κ at T_0, which is taken as the reference temperature. Also $K_\rho = \frac{1}{\rho_0}(-\frac{\partial \rho}{\partial T})_{T=T_0}\Delta T$, $K_\mu = \frac{1}{\mu_0}(-\frac{\partial \mu}{\partial T})_{T=T_0}\Delta T$, $K_\sigma = \frac{1}{\sigma_0}(-\frac{\partial \sigma}{\partial T})_{T=T_0}\Delta T$, $K_\kappa = \frac{1}{\kappa_0}(-\frac{\partial \kappa}{\partial T})_{T=T_0}\Delta T$ are the parameters measuring the rate of change with respect to temperature.

The values of K_ρ, K_μ, K_σ are > 0 for most liquids, but K_κ is positive for fluids such as water and air, where as < 0 for liquids such as lubrication oil. These type of approximation are suited well for the small temperature difference ΔT between the inclined plane and the ambient air.

Let u, v denote the components of the velocity along the x and z directions respectively, p the pressure and g denotes the acceleration due to gravity. The velocity and temperature field of the film are governed by the conservation equations for mass, momentum and thermal energy as:

$$u_x + v_z = 0 \tag{5}$$

$$\rho_0(u_t + uu_x + vu_z) = -p_x + \rho g \sin\gamma + \nabla(\mu\nabla u) \tag{6}$$

$$\rho_0(v_t + uv_x + vv_z) = -p_z - \rho g \cos\gamma + \nabla(\mu\nabla v) \tag{7}$$

$$T_t + uT_x + vT_z = \nabla(\kappa\nabla T) \tag{8}$$

where ∇ denotes the Laplacian operator. In the above Eqs. (5), (6) and (7) density ρ is taken as a constant ρ_0 in all terms except the term where it is multiplied by g, which causes the external force, according to Boussinesq approximation [6].

The pertinent boundary conditions on the plane $(z = 0)$ are:

$$u = v = 0 \tag{9}$$

$$T = T_w \tag{10}$$

and on the free surface $(z = h(x, t))$, dynamic and kinematic conditions along with Newton's law of cooling as the thermal boundary condition are:

$$\mu[(u_z + v_x)(1 - h_x^2) + 2(v_z - u_x)h_x] = (\sigma_x + h_x\sigma_z).(1 + h_x^2)^{\frac{1}{2}} \tag{11}$$

$$p_a - p + \frac{2\mu[u_x h_x^2 - (u_z + v_x)h_x + v_z]}{(1 + h_x^2)} = \sigma(T)h_{xx}(1 + h_x^2)^{-\frac{3}{2}} \tag{12}$$

$$v = h_t + uh_x \tag{13}$$

$$(T_z - h_x T_x)(1 + h_x^2)^{-\frac{1}{2}} + \frac{\kappa_g}{\kappa_T}(T - T_0) = 0 \tag{14}$$

where, p_a, κ_g and κ_T denote the atmospheric pressure, heat transfer coefficient from liquid to air and thermal conductivity of the liquid respectively.

The dimensionless variables marked by asterisk sign in the superscript are defined as:

$$x = l_0 x^*, \quad (h, z) = h_0(h^*, z^*), \quad t = \frac{l_0 t^*}{u_0}, \quad u = \frac{u_0}{u^*}, \quad v = \frac{u_0 h_0 v^*}{l_0},$$
$$p = \rho u_0^2 p^*, \quad \theta = \frac{T - T_0}{\Delta T} \tag{15}$$

Using the dimensionless variables (15) in the Eqs. (5)–(14) we arrive after dropping the asterisk as:

I. Governing equations:

$$u_x + v_z = 0 \tag{16}$$

$$\epsilon Re(u_t + uu_x + vu_z) = -\epsilon Re p_x + 3(1 - K_\rho\theta) - K_\mu(\epsilon^2 u_x\theta_x + u_z\theta_z) + (1 - K_\mu\theta)(\epsilon^2 u_{xx} + u_{zz}) \tag{17}$$

$$\epsilon^2 Re(v_t + uv_x + vv_z) = -Re p_z - 3\cot\gamma(1 - K_\rho\theta) - \epsilon K_\mu(\epsilon^2 v_x\theta_x + v_z\theta_z) + \epsilon(1 - K_\mu\theta)(\epsilon^2 v_{xx} + v_{zz}) \tag{18}$$

$$\epsilon Re Pr(\theta_t + u\theta_x + v\theta_z) = K_\kappa(\epsilon^2\theta_x^2 + \theta_z^2) + (1 + K_\kappa\theta)(\epsilon^2\theta_{xx} + \theta_{zz}) \tag{19}$$

II. Boundary conditions at the inclined plane $(z = 0)$:

$$u = v = 0 \quad \text{and} \quad \theta = 1 \tag{20}$$