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Equivalence theorem between position-dependent mass dynamics and classically conservative dynamics

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Dedicated to my father, Eny Casetta

ABSTRACT

Variable-mass problems do not as a rule fit into the cardinal formulation of mechanics; therefore, new formalism has been constructed to treat variable-mass dynamics. We aim to situate a class of position-dependent mass problems in the level of classically conservative dynamics. The issue is that, by nature, the sum of kinetic and potential energies of a position-dependent mass point is not preserved. Given that, we demonstrate a theorem which establishes the mathematical equivalence between position-dependent mass dynamics and classically conservative dynamics. Meshchersky's equation is herein assumed to be in scalar form. In applying the theorem, a counterintuitive situation arises. To our very best knowledge, our contribution is novel in the field of variable-mass dynamics.

1. Introduction

The mass of a system can be considered to be variable when matter is permitted to cross the system boundary. Although not formal, this is an easily conceived notion of the variable-mass condition. Unfamiliar readers find in [1,2] an elucidating opening about the theme. For other authors who have contributed to the foundations of the variable-mass dynamics, see [3] and references therein cited.

Turning the attention to history (see [1,4]), we can infer that the development of such branch of mechanics has been propelled by two reasons. First, one observes that there are real world problems that involve variable-mass systems. The rocket problem is an archetypical example in this sense. For other important problems, see [1–3]. Second, there is a mathematical difficulty which is inherent to the field. Namely, given that “the fundamental equations of classical mechanics were originally formulated for the case of an invariant mass contained in a material volume” (see [1, p. 145]), providing variable-mass problems with a suitable formalism then arises as a challenging issue. On this point, there is a nexus of contributions establishing the founding bases (see [1–22]).

The scope of our article is the demonstration of a theorem which translates the dynamics of a position-dependent mass system into the dynamics of a classically conservative system. To wit, attention is directed to points with mass depending only upon the generalized coordinate for which Meshchersky's equation is assumed to be in scalar form. Hence, the aimed contribution links us intimately with the motivation of sighting variable-mass problems from the Weltanschauung of classical mechanics. To our very best knowledge, this is original in the field of variable-mass dynamics.

The issue in question is that position-dependent mass problems are per se nonconservative (see [7]). Videlicet, “by nature, the sum of kinetic and potential energies of a position-dependent mass particle is not generally preserved along its motion” (see [5, pp. 351–352]). As proved in [5], it is the total energy multiplied by a given function that in truth is conserved in position-dependent mass dynamics. Given that, the novel theorem elicits the classical conservation of such variable-mass problems.

The article is organized as it follows. In Section 2, we set the stage for the discussion. In Section 3, the aimed theorem is established. In Section 4, we apply the theorem directly and inversely, wherewith examples are addressed.

2. Preliminaries

The equation of motion of a variable-mass point is Meshchersky's equation

$$m\ddot{q} - Q - (w - \dot{q})\dot{m} = 0, \tag{1}$$

where m is varying mass, q is generalized coordinate, Q is generalized force, w is absolute velocity of mass ejection (or aggregation), t is time, overdot is derivative in t. Eq. (1) entails also that q is a coordinate linear in dimension of length.

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Under the assumptions that $m = m(q)$, $Q = -dV(q)/dq$, where $V = V(q)$ is potential energy, and that $w = k\dot{q}$, where $k = const.$ (i.e., linear in $\dot{q}$); Eq. (1) yields

$$m(q)\ddot{q} + \frac{dV(q)}{dq} - \alpha\dot{q}^2\frac{dm(q)}{dq} = 0, \tag{2}$$

where $\alpha = k - 1 = const.$

The analytical formulation [10] gives us the following conservation law of Eq. (2):

$$\frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 + \int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq = \widetilde{E}, \tag{3}$$

where $\widetilde{E} = const.$ For a more detailed explanation, see [5,7,8,10].

3. Theory

We now initiate the demonstration of our contribution.

Theorem. *The dynamics of a position-dependent mass system, which herein is assumed to be governed by* Eqs. (2) and (3)*, is mathematically equivalent to the dynamics of the classically conservative system that is governed by the equations*

$$\ddot{q} + \frac{d\Phi(q)}{dq} = 0, \tag{4}$$

$$\frac{1}{2}\dot{q}^2 + \Phi(q) = E, \tag{5}$$

where $E = const.$ *and*

$$\Phi(q) = \frac{1}{m(q)^{-2\alpha}}\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq - \frac{\widetilde{E}}{m(q)^{-2\alpha}} + E. \tag{6}$$

Proof. First, we prove that Eq. (2) can be put in the form of (4).

We organize Eq. (3) into

$$\dot{q}^2 = 2m(q)^{2\alpha}\left[\widetilde{E} - \int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq\right]. \tag{7}$$

Inserting Eq. (7) in (2), we find

$$m(q)\ddot{q} + \frac{dV(q)}{dq} - \alpha\left\{2m(q)^{2\alpha}\left[\widetilde{E} - \int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq\right]\right\}\frac{dm(q)}{dq} = 0. \tag{8}$$

Dividing Eq. (8) by $m(q)$, we obtain

$$\ddot{q} + \left\{m(q)^{-1}\frac{dV(q)}{dq} - 2\alpha m(q)^{2\alpha-1}\left[\widetilde{E} - \int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq\right]\frac{dm(q)}{dq}\right\} = 0. \tag{9}$$

In Eq. (9), the term enclosed by the curly braces is a function depending only upon q. Let us suppose as an ansatz that such term is of the form $d\Phi(q)/dq$, which yields

$$\ddot{q} + \frac{d\Phi(q)}{dq} = 0, \tag{10}$$

where

$$\frac{d\Phi(q)}{dq} = m(q)^{-1}\frac{dV(q)}{dq} - 2\alpha m(q)^{2\alpha-1}\left[\widetilde{E} - \int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq\right] \times \frac{dm(q)}{dq}. \tag{11}$$

Then, having found Eq. (10), it is immediate that

$$\frac{1}{2}\dot{q}^2 + \Phi(q) = const. \tag{12}$$

Eq. (12) is a first integral of (10). For the sake of a convenient symbology, we rewrite the right-hand side of Eq. (12) as

$$E = const. \tag{13}$$

To close the proof, we have to obtain Eq. (6) in a consonant manner. Based on rules for differentiation, we write

$$\frac{d}{dq}[m(q)^{2\alpha}] = 2\alpha m(q)^{2\alpha-1}\frac{dm(q)}{dq}, \tag{14}$$

$$\frac{d}{dq}[m(q)^{2\alpha}]\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq = \frac{d}{dq}\left[m(q)^{2\alpha}\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq\right] - m(q)^{-1}\frac{dV(q)}{dq}. \tag{15}$$

Making use of Eqs. (14) and (15), we manipulate Eq. (11) into

$$\frac{d}{dq}\left[\Phi(q) - \frac{1}{m(q)^{-2\alpha}}\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq + \frac{\widetilde{E}}{m(q)^{-2\alpha}}\right] = 0, \tag{16}$$

which, after integration, gives

$$\Phi(q) = \frac{1}{m(q)^{-2\alpha}}\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq - \frac{\widetilde{E}}{m(q)^{-2\alpha}} + C, \tag{17}$$

where $C = const.$

Now it remains only to show that $C = E$. Looking at Eq. (3), we manipulate it into

$$\frac{1}{2}\dot{q}^2 + \frac{1}{m(q)^{-2\alpha}}\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq - \frac{\widetilde{E}}{m(q)^{-2\alpha}} = 0. \tag{18}$$

Comparing Eqs. (12) and (13) with (18), we have

$$\Phi(q) = \frac{1}{m(q)^{-2\alpha}}\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq - \frac{\widetilde{E}}{m(q)^{-2\alpha}} + E. \tag{19}$$

And, comparing Eqs. (17) and (19), we then obtain

$$C = E. \tag{20}$$

The proof is concluded. □

Established the equivalence, the following corollary is immediate:

Corollary. *It naturally follows from the theorem that, if* Eq. (6) *is satisfied, the solution of the equation of motion (4) equals the solution of the equation of motion (2).*

Remark 1. To show the relationship between the integration constants $\widetilde{E}$ and E, we particularize Eq. (6) for some initial condition $q(t = 0) \equiv q_0$:

$$\Phi(q_0) - E = \frac{1}{m(q_0)^{-2\alpha}}\left[\int m(q)^{-2\alpha-1}\frac{dV(q)}{dq}dq\right]_{q=q_0} - \frac{\widetilde{E}}{m(q_0)^{-2\alpha}}. \tag{21}$$

4. Application

Eq. (6) thus relates the position-dependent mass problem, specified by $m(q)$, $V(q)$, α, and $\widetilde{E}$, to the classically conservative problem, specified by $\Phi(q)$ and E.

Direct method. *Let us assume that the functions $m(q)$ and $V(q)$, the constant α, and the constant $\widetilde{E}$ are given. Then, the quantity $\Phi(q) - E$ is calculated via* Eq. (6)*. Namely, given the position-dependent mass problem, which herein is the real problem, we therefrom go in search of the equivalent conservative problem.*

Example 1. The classical Cayley's falling-chain problem [14, p. 506] is such that "(...) a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table; the part hanging over constitutes the moving system (...)".

Following [8,10], we recall $m(q)$, $V(q)$, α, $\widetilde{E}$, and initial conditions; i.e. since particles to be captured by the moving system are at rest, then $w = 0$, which renders[1]

$$\alpha = -1. \tag{22}$$

The coordinate q of the lower extremity of the chain is the generalized coordinate, and the mass of the moving part is

$$m(q) = \rho q, \tag{23}$$

[1] For the definition of alpha, see Eq. (2).