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Conservation laws and conserved quantities of the governing equations for the laminar wake flow behind a small hump on a solid wall boundary

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ABSTRACT

The conservation laws and conserved quantities for the governing equations of the two-dimensional laminar wake flow behind a hump on a flat plate are derived. The multiplier method is applied to the linearised governing equations for small humps and a basis of conserved vectors is constructed. Since, in general, the problem contains an unknown non-homogeneous boundary condition, each conserved vector needs to be carefully chosen and additional restrictions need to be applied to ensure that each conserved quantity, which is obtained by integrating the corresponding conservation law across the wake and imposing the relevant boundary conditions, has a finite value. Four non-trivial conserved quantities are found; three of which have only now been identified. The four conserved quantities relate to the conservation of mass, drag and the first and second moments of the momentum deficit. For each case the existence of a solution that satisfies the governing equations, boundary conditions and a finite valued conserved quantity is discussed.

1. Introduction

The governing equations for the two-dimensional laminar wake flow behind a hump situated on a solid wall boundary, also known as the laminar 'wall-wake', are examined. The problem of the wall-wake was first proposed by Hunt [1]. A boundary layer is perturbed by a small hump on an otherwise flat plate. Hunt's [1] approach was to divide the flow behind the hump into two regions: an inner viscous flow region near to the wall and an intermediate inviscid region that matches to the unperturbed boundary layer flow. The wake flow is contained within the inner viscous flow region [1]. A further investigation into wall-wake flows using a different approach was performed by Smith [2]. Smith [2] applied triple deck theory [3,4], which has proved to be very successful in describing perturbed boundary layer flows, to the problem of the wall-wake. In addition to the two main regions or decks of flow that Hunt [1] defined, Smith [2] identified a third deck of inviscid flow outside of the boundary layer. This third deck is required because the flow outside of the boundary layer is displaced by the presence of the hump [2]. This is known as the boundary layer displacement effect.

At first appearance the results by Hunt [1] and Smith [2] are contradictory. Upon further investigation however, the results can be reconciled by applying triple deck theory which considers three main regions or decks of flow [5]. It was argued that Hunt's approach [1] solved for the near wake on the triple deck scale where only two decks are needed because the boundary layer displacement effect is negligible in this case [5]. Smith's solution [2] described the far wall-wake on the triple deck scale where all three decks are needed in order to include the boundary layer displacement effect [5]. For both the near and far wall-wakes, the wake is confined to the lower deck which is bounded on one side by the flat plate. The governing equations for the wake are solved subject to the no-slip condition at the solid wall interface, the matching conditions between the lower and intermediate decks which differ for near and far wakes, and if applicable, a conserved quantity. Inclusion of the boundary layer displacement effect results in a non-homogeneous boundary condition at the interface between the lower and intermediate decks. In the case of the near wake where the function describing the boundary layer displacement effect is set to zero, the boundary conditions between the lower and intermediate decks are homogeneous.

The governing equations for the wall-wake are non-linear. However, for very small humps, the governing equations can be linearised [2,5]. When the boundary layer displacement effect is included, the governing equations and the boundary conditions are, in general, not homogeneous [2]. For the far wall-wake, the boundary layer displacement effect is specified which determines the non-homogeneous boundary condition [2]. Because the governing equations and boundary conditions are not homogeneous, a conserved quantity is not needed to complete

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the solution [2,5]. For the near wall-wake, because the boundary layer displacement effect is negligible, the governing equations and boundary conditions are homogeneous and a conserved quantity is required to complete the solution [1]. For both the near and far wall-wakes, the boundary layer displacement effect is specified. If, however, the boundary layer displacement effect is not known then the governing equations need to be solved subject to an unknown non-homogeneous boundary condition. There is insufficient knowledge on this problem in the current literature to ascertain as to whether a conserved quantity is required to complete the solution when the boundary layer displacement effect is unknown. The aim of the present work is to adapt and apply existing theory on conservation laws and conserved quantities to the governing equations of the wall-wake in order to derive a basis for the conserved vectors and to determine the conditions for which a finite conserved quantity corresponding to each conservation law exists. If the boundary layer displacement effect is not specified which then results in a non-homogeneous boundary condition, it is shown that under certain conditions finite conserved quantities can be found and that the boundary layer displacement effect can be determined.

In [6], various approaches to finding the conservation laws for partial differential equations are discussed. Once a conserved vector has been obtained, the Lie symmetry associated with this conserved vector can be calculated and then used to generate the invariant solution [7,8]. For problems where a conserved quantity is required to complete the solution, the double reduction theorem can be used [9]. Other works on symmetries and conservation laws for differential equations are given in [10–22]. In this work the multiplier method [10,23] is used to calculate a basis of conserved vectors for the governing equations of the wall-wake when expressed in terms of the velocity components and when expressed in terms of the stream function. This method has been used to calculate the conservation laws for the radial and two dimensional free jets [24] and for the classical wake and the wake of a self-propelled body [25]. For the governing equations pertaining to the wall-wake problem, four conservation laws are obtained. One of these corresponds to the near wall-wake whilst the rest are newly discovered. Each conservation law is then integrated across the wake and the relevant boundary conditions are imposed in order to generate the required conserved quantity. Much consideration needs to be taken when deriving the conserved vectors. As there is a possibility of an unknown non-homogeneous boundary condition, convergence of the integrals arising from integrating a conservation law across the wake is not guaranteed. However, it is shown how this issue can be overcome. The conserved quantity for the near wall-wake, which is the moment of momentum deficit, is re-derived in a systematic way. It is discovered that each of the three remaining conserved quantities correspond to the conservation of mass, drag and the second moment of the axial momentum deficit.

This paper is outlined as follows. In Section 2, a detailed description of the mathematical model is provided and the governing equations for small humps along with the boundary conditions are given. In Section 3 the general theory for the multiplier method is presented. It is discussed how conserved vectors are chosen for problems with unknown non-homogeneous boundary conditions. The conservation laws for the governing equations for the wall-wake are derived in terms of the velocity components in Section 4.1 and in terms of the stream function in Section 4.2. In Section 5 the conservation laws are integrated across the wake to obtain the conserved quantities. Additional conditions that need to be imposed in order to generate finite conserved quantities are discussed. The conserved quantities in terms of the velocity components are given in Section 5.1 and in terms of the stream function in Section 5.2. In Section 5.3 a summary of the findings on conserved vectors is given including the requirements for the corresponding conserved quantity to exist. The physical significance of each conserved quantity is analysed in Section 5.4. In Section 6 similarity solutions of the governing equations are studied. Invariance of each conserved quantity enables the form of the similarity solution to be identified. The similarity solutions are then solved and it is shown that finite conserved quantities can be obtained. Conclusions are given in Section 7.

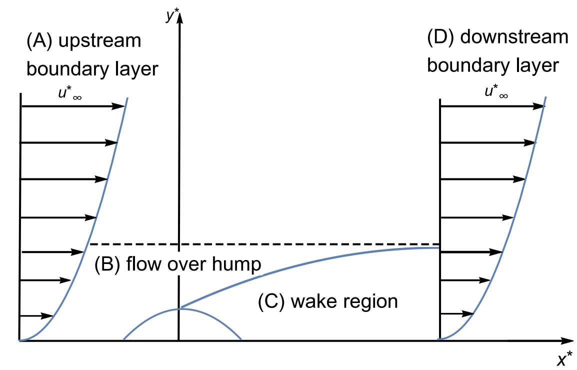

Fig. 1. Stages in a wall-wake flow.

2. Mathematical model

Consider a laminar stream of viscous incompressible fluid flowing past a small symmetric hump on an otherwise smooth boundary. A Cartesian coordinate system (x^*, y^*) is used. The line $y^* = 0$ lies along the solid wall boundary and the line $x^* = 0$ lies along the axis of symmetry of the hump. The constant mainstream speed, density and kinematic viscosity of the fluid are given by u^*_∞, ρ, and $\nu = \mu/\rho$ respectively, where μ is the dynamic viscosity. The flow transitions through four different stages as shown in Fig. 1. In stage A, the far upstream boundary layer flow is unaffected by the presence of the hump. Stage B represents the flow over the hump and the flow near to the leading and trailing edges of the hump. Once the boundary layer flow comes into contact with the hump, it is perturbed and a wake forms directly downstream of the hump as shown in stage C. Sufficiently far downstream, the flow reverts to its upstream configuration as shown in stage D. In this paper, particular attention is paid to the governing equations that describe the wake flow region.

Triple deck theory can be used to derive the governing equations for the wall-wake flow for both the near and far wall-wakes which satisfy the same governing equations, but different boundary conditions [2,5]. The near wake flow applies for small x^* and the far wake flow is relevant for large x^*. The x^*- and y^*- velocity components and the fluid pressure in the wake are denoted by $u^*(x^*, y^*)$, $v^*(x^*, y^*)$ and $p^*(x^*, y^*)$ respectively. The Reynolds number Re for the flow is defined in terms of the upstream boundary layer flow [2]:

$$Re = \frac{u^*_\infty L}{\nu}, \tag{2.1}$$

where L is the development length of the oncoming boundary layer which determines the boundary layer thickness $\delta = LRe^{-\frac{1}{2}}$. The implementation of triple deck theory to this problem relies on the assumption that the parameter, ϵ, where [3]

$$\epsilon = Re^{-\frac{1}{8}}, \tag{2.2}$$

is small which is true for very large Reynolds numbers. The flow is further divided into three sub-regions or decks as shown in Fig. 2.

Dimensionless variables are defined as follows:

$$x^* = \epsilon^n Lx, \qquad y^* = \epsilon^m Ly,$$
$$u^* = u_\infty u, \qquad v^* = u_\infty v, \qquad p^* = p^*_\infty + \rho u^{*2}_\infty p, \tag{2.3}$$

where n and m are positive integers. Substituting (2.3) into the Navier–Stokes equation and the continuity equation results in

$$\epsilon^{m-n} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\epsilon^{m-n}\frac{\partial p}{\partial x} + \epsilon^{8+m-2n}\frac{\partial^2 u}{\partial x^2} + \epsilon^{8-m}\frac{\partial^2 u}{\partial y^2}, \tag{2.4}$$

$$\epsilon^{m-n} u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \epsilon^{8+m-2n}\frac{\partial^2 v}{\partial x^2} + \epsilon^{8-m}\frac{\partial^2 v}{\partial y^2}, \tag{2.5}$$

$$\epsilon^{m-n}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.6}$$