ARTICLE IN PRESS

International Journal of Non-Linear Mechanics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

# Bifurcations in the axial–torsional state-dependent delay model of rotary drilling

Sunit K. Gupta *, Pankaj Wahi

*Mechanical Engineering Department, Indian Institute of Technology Kanpur, UP, 208016, India*

ARTICLE INFO

*Keywords:*
Rotary drilling
State-dependent delay
Method of multiple scales
Bifurcation analysis
Double Hopf point

ABSTRACT

We present a detailed bifurcation analysis of the state-dependent delay model of rotary drilling considering only the axial and torsional modes. This analysis is presented for the general case of independent natural frequencies of these two modes. The regenerative effect accompanying axial vibrations gives rise to a delayed model with the delay determined by the torsional oscillations. It is observed that steady drilling loses stability through a Hopf bifurcation. The nature of bifurcation is ascertained by the method of multiple scales for the general values of system parameters. Analytical results suggest that both supercritical and subcritical bifurcations exist for different choices of operating and system parameters. These analytical findings are further confirmed by numerical simulations. Possible unfoldings of the dynamics near the codimension-2 point, guided by numerical simulations and analytical results for the codimension-1 Hopf branches, are also presented. We find two different scenarios at the primary codimension-2 point viz. both Hopf branches having supercritical bifurcation, and one branch being supercritical while the other being subcritical. Our numerical simulations suggest that the dynamics near the codimension-2 point is dominated by the low-frequency limit cycles in both the scenarios.



## 1. Introduction

Self-excited vibrations of drill-strings in axial, torsional and lateral directions are one of the possible reasons for their failure and damages to the hole walls [1,2]. Therefore, it is necessary to understand the dynamics and the coupling of various modes of drill-strings during these self-excited vibrations. A lumped-parameter model including only the first axial and torsional modes of the drill-string coupled via the regenerative effect [3–7] is one of the preferred models for such studies. Bifurcations in the state-dependent delay model of [4] for the special case of 1 : 1 internal resonance between the axial and torsional modes has recently been presented by Gupta and Wahi [8]. The analysis in [8] for this special case reveals the possibility of both subcritical and supercritical bifurcations. However, the analysis in [9] showed that the stability boundaries for the general untuned system is qualitatively different from the tuned system. Hence, a systematic analysis of the bifurcation characteristics for the general case is required to fully comprehend the system behavior. This analysis has been presented in the current work where we have used the method of multiple scales (MMS) for the analytical treatment of the simple Hopf bifurcations and numerically studied the dynamics close to the double Hopf point.

The 'regenerative effect' was identified as one of the sources of excessive tool vibration during machining by Tlusty [10] and Tobias [11]. It is associated with a wavy cut surface on the work-piece due to relative vibrations between the tool and the work-piece which causes a chip-thickness variation during the next pass of the tool. This results in fluctuating cutting forces which further excites the vibrations leading to a feedback loop. This effect is known as the regenerative effect and the corresponding model turns out to be a delay differential equation since the current chip thickness depends on the previous path of the tool. The simplest model for regenerative machine tool vibrations is a constant delay differential equation (CDDE). Experimental studies on machine tool vibration [12–14] and nonlinear analysis of the CDDE model using analytical methods [12,13,15–18] have shown a subcritical Hopf bifurcation only. Later on, Insperger et al. [19] noted that regenerative effect in machining operations can be more accurately modeled by a state-dependent delay differential equation (SDDDE) by including tangential vibrations of the tool. The presence of tangential vibrations leads to an implicit dependence of the delay on the tangential oscillations of the tool.

A regenerative axial–torsional model for rotary drilling was probably developed for the first time by Richard et al. (RGD model) [3]. In this

* Corresponding author.
*E-mail address:* sunitgpt@iitk.ac.in (S.K. Gupta).

https://doi.org/10.1016/j.ijnonlinmec.2017.10.018
Received 22 September 2017; Accepted 21 October 2017
Available online xxxx


Please cite this article in press as: S.K. Gupta, P. Wahi, Bifurcations in the axial–torsional state-dependent delay model of rotary drilling, International Journal of Non-Linear Mechanics (2017), https://doi.org/10.1016/j.ijnonlinmec.2017.10.018.

model, the axial motion results in the regenerative effect while the torsional motion is responsible for the state-dependency of the delay. Hence, the simplest model for regenerative rotary drilling applications is necessarily a SDDDE. The primary purpose of the RGD model [3] was to understand the origin of stick–slip oscillations in drill-strings. We note that the RGD model did not incorporate any damping and axial stiffness, and stable steady drilling is not possible for any choice of operating parameters [9,20]. Following on the work of Richard et al. [3], Besselink et al. [21] and Nandakumar and Wiercigroch [4] included axial stiffness and damping in the axial and torsional directions in their models. These modifications in the RGD model led to some region in the operating parameter space for stable steady drilling as elucidated in the subsequent analysis [6,7,9,22]. It has to be noted that these studies [6,7,9,22] on the SDDDE model of rotary drilling focused on the linear stability and the nonlinear effect of the state-dependency of the delay was not discussed.

There is limited literature available on the nonlinear analysis of SDDDEs. The first such study to the best of our knowledge was performed by Insperger et al. [23] on the SDDDE model of turning [19]. This was performed using the numerical continuation package DDEbiftool wherein the authors observed that the nature of Hopf bifurcation can change from subcritical to supercritical because of state-dependency of the delay. This observation was further confirmed by analytical findings using MMS [24]. It is noted that both these studies were performed for the special case of 1:1 internal resonance between the vibration modes in the two directions for which the 2-degree of freedom mechanical model reduces to a single-degree of freedom. Wahi [24], in his analytical study, converted the SDDDE near the Hopf point into a perturbed DDE with a constant delay by expanding the state-dependent delay in a series of some appropriate small parameter and then applied the MMS procedure for DDE outlined in [25–28]. Following this analytical approach, Kim and Seok [29] presented a nonlinear analysis of the general SDDDE model of turning using MMS. However, the authors did not specify the exact procedure of getting the slow flow equations and the numerical results are shown only for the special case of 1:1 resonance between the two modes. Application of MMS for coupled DDEs in [28] emphasized the fact that special procedure is required for removal of secular terms which leads to the slow flow equations in these cases and hence, the correctness of the results of [29] could not be verified in the absence of these details.

A detailed bifurcation analysis of rotary drilling with varying operating parameters was recently presented in [5] for a particular choice of system parameters. This study incorporated the global nonlinearities associated with bit-bounce (loss of contact between the drill-bit and the cut surface) and stick–slip (no rotation of the drill-bit during periods of motion). Incorporation of these effects necessitated development of an alternate model for the regenerative cutting mechanism along the lines of [30]. It is to be noted that for small amplitude vibrations wherein there is no bit-bounce and stick–slip motions, this global model [5] is exactly the same as the SDDDE model as was observed from the linear stability analysis presented in [9]. Bifurcation analysis for the special case of 1:1 internal resonance between the axial and torsional modes was recently presented in [8] wherein MMS was applied to the reduced single-degree of freedom mechanical model along the lines of [24]. Both subcritical and supercritical bifurcations were obtained followed by fairly complicated dynamics which were unveiled numerically. We note from the study in [9] that the stable regions for the general axial–torsional model of rotary drilling is different from the special case of the 1:1 internal resonance between the two modes. Hence, to fully understand the nonlinear dynamics associated with the general case of coupled SDDDE axial–torsional model of rotary drilling, we present a detailed bifurcation analysis using MMS and numerical simulations in this paper. The analytical approach using MMS in this paper is a combination of that presented in [8] (for the state-dependent delay term) and [28] (to handle the higher dimensionality in the mechanical model). We again observe both supercritical and subcritical bifurcations in different portions of the stability boundary. Changes in the supercritical region while varying system parameters leading to interesting scenarios near the codimension-2 points has also been presented followed by a numerical unfolding of the dynamics around this point.

Rest of the paper is organized as follows. We present a brief outline of the development of the SDDDE model of rotary drilling in Section 2 along with its representation as a perturbed CDDE for further analysis. Linear stability analysis of the equilibrium is presented in Section 3. In Section 4, nonlinear analysis of the system using MMS is presented followed by a discussion of these results in Section 5. Section 6 presents numerical bifurcation results to verify the analytical results of MMS and unfold the dynamics near the codimension-2 point. Finally, some conclusions are drawn in Section 7.

## 2. The mathematical model

In this section, we briefly present the SDDDE model of drill-string which is employed for the current analysis and its reduction to the perturbed CDDE model [8]. We have considered the drill-string as a two degree-of-freedom spring–mass–damper system for the axial and torsional motions [4,5] as shown in Fig. 1. We have assumed that the top of the drill-string moves with a constant feed-velocity of $V_0$ (similar to that in [4,6,21,22]) which is an idealization for the boundary condition at the top. The true boundary condition at the top surface involves an equivalent spring–mass–damper system to model the traveling block and the hoisting cable [31–35] which adds to the complexity of the model. The alternate idealization of a constant force at the top results in the RGD model with no axial flexibility [3,36] which can be obtained as a special case of the model considered in the present work.

With the assumption of the top of the drill-string moving with a constant velocity $V_0$, the equations of motion corresponding to the axial and the torsional modes can be written as

$$M\ddot{U}(t) + C_a\dot{U}(t) + K_a\left\{U(t) - V_0 t\right\} = W_0 - F_c\,, \tag{1a}$$

$$J\ddot{\Phi}(t) + C_t\dot{\Phi}(t) + K_t\left\{\Phi(t) - \Omega_0 t\right\} = -T_c\,, \tag{1b}$$

where $\Omega_0$ is the angular velocity at the top of the drill-string, $W_0$ is the net weight on the drill-bit, $K_a$ and $K_t$ are the spring stiffnesses in axial and torsional directions, respectively, $C_a$ and $C_t$ are viscous damping coefficients in axial and torsional directions, respectively, $M$ and $J$ are the combined mass and rotary inertia about the rotational axis of the drill-string and the bottom hole assembly (BHA), respectively, and $F_c$ and $T_c$ represent the force and torque because of the cutting action, respectively. Note that we have ignored any wear flat on the drill-bit and consequently there are no frictional forces and torques. The cutting force and torque on the drill-bit are related to the system and operational parameters as [3],

$$\begin{aligned} F_c &= \xi\epsilon a d(t) H(\dot{\Phi}) H(d(t)),\\ T_c &= \frac{\epsilon a^2 d(t)}{2} H(\dot{\Phi}) H(d(t)), \end{aligned} \tag{2}$$

where $H(.)$ represents the Heaviside step function, $\xi$ is the cutter inclination coefficient, $\epsilon$ is the rock specific strength, $a$ is the radius of the drill-bit, and $d(t)$ is the instantaneous depth of cut per revolution of the drill-bit. Assuming that the rock is homogeneous and the drill-bit has $n$ identical cutters, the total depth of cut per revolution can be written as

$$d(t) = n d_n(t)\,, \tag{3}$$

with $d_n(t)$ as the depth of cut per cutter (see Fig. 1(c)) given by [3,4]

$$d_n(t) = U(t) - U(t - t_n). \tag{4}$$

In the above, $t_n$ is the time taken by the drill-bit to rotate by an angle of $2\pi/n$ which can be computed through

$$\Phi(t) - \Phi(t - t_n) = \frac{2\pi}{n}. \tag{5}$$



Please cite this article in press as: S.K. Gupta, P. Wahi, Bifurcations in the axial–torsional state-dependent delay model of rotary drilling, International Journal of Non-Linear Mechanics (2017), https://doi.org/10.1016/j.ijnonlinmec.2017.10.018.