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Two-dimensional waves in extended square lattice

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ABSTRACT

We consider a two-dimensional square lattice model extended by additional not closed neighboring interactions. We assume the elastic forces between the masses in the lattice to be nonlinearly dependent on the spring elongations. First, we use an analysis of the linearized discrete equations to reveal the influence of additional interactions on the properties of the dispersion relation for longitudinal and shear plane waves. Then we develop an asymptotic procedure to obtain continuum two-dimensional non-linear equations to study the transverse instability of weakly non-linear localized plane longitudinal and shear waves. We find that the additional interactions used in the model may affect the sign of the amplitude of the plane strain waves (existence of compression (minus sign) or tensile (plus sign) plane waves) and their transverse stability.

0. Introduction

The study of the discrete model with non-neighboring interactions between the particles in the lattice has attracted considerable interest due to the dispersion of waves propagating in such a system [1–8]. In particular, this model is important for the study of the influence of the microstructure of materials. Dynamic processes in one-dimensional lattices have been investigated more extensively [1,3,9], while two-dimensional lattices are mainly considered in the linearized case [3,6,7]. Some two-dimensional processes can be modeled in the one-dimensional approximation, like plane waves propagation, while the study of their transverse instability requires two-dimensional consideration. Also some physical phenomena cannot be modeled in the one-dimensional case, in particular, for a negative Poisson ratio or auxetic behavior [10–13].

The structural features of the lattice are usually taken into account [10,14,15] to describe a negative Poisson ratio. In [11] it was obtained that a negative Poisson ratio is observed for some directions in many cubic metals due to their crystalline lattice features. It is also known that anisotropic systems like cubic ones are typically nonauxetic or partially auxetic [16]. The relationships for an anisotropic Poisson ratio in some materials may be found in [17,18]. There is a procedure for comparing the continuum limits of 2D discrete models with the 2D limit of the continuum cubic crystal model [15] to establish a connection between the rigidities of the lattice model and the cubic elastic constants. It turns out that these relationships hold only for the Cauchy condition [19]. It applies to materials with a cubic symmetry where only central interactions are taken into account; however, deviations from the conditions may be considerable, e.g., for cubic metals [20]. However, it was found in [21] that the Cauchy relations do not hold for positive temperatures. Comparison with the 2D model, e.g., the auxetic properties of 2D media, were studied in [22].

Dynamic processes in lattices have been studied using both discrete and continuum modeling [1,9]. In the linear case, both discrete and continuum equations can be solved analytically. However, only a few discrete non-linear equations, such as the Toda lattice equation or the Ablowitz–Ladik equation, possess exact solutions [23]. That is why an approach based on the continuum limit of the original discrete equation is needed to obtain the governing non-linear continuum equations. The familiar acoustic branch continuum limit [1,9] requires the long wavelength approximation and corresponds to the discrete model only for small wave numbers.

The mechanical properties and stability of lattices depend on their structure and particle interaction [19,24,25]. Discrete and continuum models both possess analytical solutions in the linear case, which allows complex analysis of the mechanical phenomena from micro- and macroscopic points of view [26]. This analysis becomes crucial for nano-objects where the discreteness of the atomic structure cannot be neglected [27]. Nonlinearity is essential for a description of thermo-mechanical effects [28] including peculiarities such as negative thermal expansion [29].

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Localized non-linear strain wave propagation with a permanent shape and velocity and its amplification are of special importance. The plane waves can be described within the one-dimensional model while their transverse instability and inclined waves interaction require two-dimensional consideration [2,23,30–32]. It allows us to model new types of the wave amplification and localization due to a transverse instability [23,32,33] or interaction of the plane waves [30,34,35].

In this paper, an extended two-dimensional square lattice model is considered with the addition of the nearest neighbors interactions of the central particle. The model also includes a quadratic and a cubic nonlinearity in the elastic inter-particle forces. Linearized analysis is used to study the features of the dispersion relation caused by the inclusion of the extended interactions on the basis of a plane wave approximation. Further, an asymptotic solution is developed to obtain the continuum non-linear governing equations for both longitudinal and shear plane strain waves disturbed in the direction perpendicular to their direction of propagation. The influence of the long-range interactions on a transverse instability of both types of plane waves is studied to see whether a two-dimensional localized non-linear wave can appear from localized input or is due to a resonant plane waves interaction.

1. Statement of the problem

Let us consider a square lattice discrete structure with the particles having equal masses M, see Fig. 1. One can distinguish three kinds of interaction in contrast to the two used for the standard model. That is why we call it an *extended square lattice model*. The central particle with the number m,n interacts with four horizontal and vertical neighbors by the springs with linear rigidity C_1 and non-linear rigidities Q and Q_3. The relative distance in the unstrained state is assumed to be equal to l. The contribution to the potential energy is

$$\Pi_1 = \frac{1}{2}C_1 \sum_{i=1}^{4} \triangle l_i^2 + \frac{1}{3}Q \sum_{i=1}^{4} \triangle l_i^3 + \frac{1}{4}Q_3 \sum_{i=1}^{4} \triangle l_i^4,$$

where $x_{m,n}$, $y_{m,n}$ are the horizontal and vertical displacements of particle m,n. The expressions for the elongations of the springs, $\triangle l_i$ are

$$\triangle l_1 = x_{m+1,n} - x_{m,n}, \quad \triangle l_2 = y_{m,n+1} - y_{m,n},$$
$$\triangle l_3 = x_{m,n} - x_{m-1,n}, \quad \triangle l_4 = y_{m,n} - y_{m,n-1}$$

where the springs are numbered counter-clockwise. The next group of interacting particles is composed by four diagonal neighboring particles whose positions are described by the angles $\phi = \pi/4 + \pi\, k/2,\ k = 0,\ldots,3$. The linear rigidity of the connecting springs is C_2 while the non-linear rigidities are P and P_3. The contribution to the potential energy is

$$\Pi_2 = \frac{1}{2}C_2 \sum_{i=5}^{8} \triangle l_i^2 + \frac{2\sqrt{2}}{3}P \sum_{i=5}^{8} \triangle l_i^3 + P_3 \sum_{i=5}^{8} \triangle l_i^4,$$

$$\triangle l_5 = \frac{1}{\sqrt{2}}\left(x_{m+1,n+1} - x_{m,n} + y_{m+1,n+1} - y_{m,n}\right),$$

$$\triangle l_6 = \frac{1}{\sqrt{2}}\left(x_{m,n} - x_{m-1,n+1} + y_{m-1,n+1} - y_{m,n}\right),$$

$$\triangle l_7 = \frac{1}{\sqrt{2}}\left(x_{m,n} - x_{m-1,n-1} + y_{m,n} - y_{m-1,n-1}\right),$$

$$\triangle l_8 = \frac{1}{\sqrt{2}}\left(x_{m+1,n-1} - x_{m,n} + y_{m,n} - y_{m+1,n-1}\right).$$

The final group consists of eight particles whose positions are characterized by the angles ψ, ξ, so as $\tan\psi = 1/2$, $\tan\chi = 2$. Then the elongations are

$$\triangle l_9 = \cos(\psi)(x_{m+2,n+1} - x_{m,n}) + \sin(\psi)(y_{m+2,n+1} - y_{m,n}),$$

$$\triangle l_{10} = \cos(\chi)(x_{m+1,n+2} - x_{m,n}) + \sin(\chi)(y_{m+1,n+2} - y_{m,n}),$$

$$\triangle l_{11} = \cos(\chi)(x_{m,n} - x_{m-1,n+2}) + \sin(\chi)(y_{m-1,n+2} - y_{m,n}),$$

$$\triangle l_{12} = -\cos(\psi)(x_{m,n} - x_{m-2,n+1}) + \sin(\psi)(y_{m-2,n+1} - y_{m,n}),$$

$$\triangle l_{13} = \cos(\psi)(x_{m,n} - x_{m-2,n-1}) + \sin(\psi)(y_{m,n} - y_{m-2,n-1}),$$

$$\triangle l_{14} = \cos(\chi)(x_{m,n} - x_{m-1,n-2}) + \sin(\chi)(y_{m,n} - y_{m-1,n-2}),$$

$$\triangle l_{15} = \cos(\chi)(x_{m+1,n-2} - x_{m,n}) + \sin(\chi)(y_{m,n} - y_{m+1,n-2}),$$

$$\triangle l_{16} = \cos(\psi)(x_{m+2,n-1} - x_{m,n}) + \sin(\psi)(y_{m,n} - y_{m+2,n-1}).$$

while the contribution to the energy is

$$\Pi_3 = \frac{1}{2}C_3 \sum_{i=9}^{16} \triangle l_i^2 + \frac{5\sqrt{5}}{3}S \sum_{i=9}^{16} \triangle l_i^3 + \frac{25}{4}S_3 \sum_{i=9}^{16} \triangle l_i^4,$$

where C_3 is the linear rigidity, and S and S_3 are the non-linear rigidities.

Then the total potential energy is

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3,$$

and the kinetic energy is

$$T = \frac{1}{2}M\left(\dot{x}_{m,n}^2 + \dot{y}_{m,n}^2\right).$$

Then the Lagrangian, $L = T - \Pi$, can be composed, and the Hamilton–Ostrogradsky variational principle applied to obtain the discrete governing equations of motion.

2. Linear analysis

In this Section the influence of the extended interactions on the discrete dispersion relation is studied using plane waves as an example. Also a linearized long-wave continuum limit is compared with the model of a cubic crystalline lattice to see whether extended interactions can affect the auxetic features of the continuum model.

The linearized equations of motion (when $P = P_3 = Q = Q_3 = S = S_3 = 0$) obtained from the variational principle are further reduced when the plane waves propagating in horizontal direction are studied. In this case no variation in n happens, and the equations of motion are

$$M\ddot{x}_m - \left(C_1 + C_2 + \frac{2C_3}{5}\right)(x_{m+1} - 2x_m + x_{m-1}) - \frac{8}{5}C_3(x_{m+2} - 2x_m + x_{m-2}) = 0, \tag{1}$$

$$M\ddot{y}_m - \left(C_2 + \frac{8C_3}{5}\right)(y_{m+1} - 2y_m + y_{m-1}) - \frac{2}{5}C_3(y_{m+2} - 2y_m + y_{m-2}) = 0. \tag{2}$$

2.1. Longitudinal plane waves

The longitudinal wave solution to Eqs. (1), (2) is sought in the form

$$x_{m,n} = A\exp(\imath(k_x\, l\, m - \omega\, t)),\ y_{m,n} = 0. \tag{3}$$

It gives rise to the dispersion relation,

$$\omega^2 = \frac{4\sin^2\left(\frac{k\,l}{2}\right)(5C_1 + 5C_2 + 16C_3\cos(k\,l) + 18C_3)}{5M}. \tag{4}$$

First, it follows from Eq. (4) that the wave velocity is always higher in the extended case than in the standard case ($C_3 = 0$). Also the shape of the curve for ω^2 may contain more maxima–minima in the extended case, see Fig. 2. Then the phase velocity varies in $k\,l$ different from the velocity in the standard case as shown in Fig. 3. In particular, there may be an increase in the velocity at some interval, see dashed line in Fig. 3.

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