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Fractional conformal invariance method for finding conserved quantities of dynamical systems

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ABSTRACT

For a dynamical system that can be transformed into fractional Birkhoffian representation, under a more general kind of fractional infinitesimal transformation of Lie group, we present the fractional conformal invariance method and it is found that, using the new method, we can find a new kind of non-Noether conserved quantity; and we find that, as a special case, an autonomous fractional Birkhoffian system possesses more conserved quantities. Also, as the fractional conformal invariance method's applications, we, respectively, explore the conformal invariance and conserved quantities of a fractional Lotka biochemical oscillator and a fractional Hojman–Urrutia model. This work constructs a basic theoretical framework of fractional conformal invariance method, and provides a general method for finding conserved quantities of an actual fractional dynamical system that is related to science and engineering.

1. Introduction

The research on invariants of a dynamical system possesses important theoretical and practical significance, and plays an important role in nonlinear science, engineering science, mathematics, mechanics and physics. Arnold even thinks [1]: invariants are not only the foundation of Hamiltonian flow, but also as a basis for the whole of mechanics. The invariants of a dynamical system have a close relation with its symmetry, and the invariants can better reveal the profound internal properties and dynamical behaviors of a dynamical system. The methods of finding invariants by using symmetries are mainly: Noether symmetrical method [2], Lie symmetrical method [3] and Mei symmetrical method [4]. In 1997, Galiullin et al. studied conformal invariance i.e., conformal symmetry, of the Birkhoffian system under a special infinitesimal transformation [5]. In fact, the conformal invariance is a fundamental and important symmetrical method and under certain conditions, it also can lead to conserved quantities. Over the past 10 years, the study of conformal invariance has become a hot topic and been applied in Lagrangian system [6–8], Hamiltonian system [9–11], generalized Hamiltonian system [12], nonholonomic mechanical system [13–17], Birkhoffian system [18], dynamical system of relative motion [19], variable mass system [20,21], mechanico-electrical system [22], dynamical system of thin elastic rod [23,6], and so on. But, the study of conformal invariance still has been limited to the integer-order dynamics level.

Fractional dynamical method can more truly reveal the internal structure and dynamical behaviors of a dynamical system, and it is more close to the natural phenomena. Over the past 40 years, scientists began to study many problems about the dynamical system with fractional derivatives. Also the study of fractional dynamics has become a hot topic and won wide development, which includes the fractional Lagrangian and Hamiltonian mechanics [24–34], the fractional dynamics of nonholonomic systems [34], the fractional generalized Hamiltonian mechanics [35–47] and the fractional Nambu dynamics [48,47]. In fact, the Birkhoffian system is more general than the Hamiltonian system, and it is a kind of important and basic dynamical system [49–55]. Recently, the authors of Refs. [56–63,47,48] explored the fractional Birkhoffian system, also revealed its internal properties and dynamical behaviors. However, for a dynamical system that can be transformed into fractional Birkhoffian representation, the fractional conformal invariance method for finding conserved quantities is not presented. How can we present fractional conformal invariance method of a fractional Birkhoffian system? How can we find conserved quantities of a dynamical system by fractional conformal invariance? It is a very fundamental, important and interesting problem which is explored and solved in this paper.

In this paper, we present a new method of fractional dynamics, i.e., the fractional conformal invariance method, and explore the conserved quantities led by the fractional conformal invariance. Also we study its applications.

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Section 2 explains briefly the fractional Birkhoffian system, and provides a fractional Birkhoffian method of constructing a family of fractional dynamical models.

In Section 3, we introduce a more general kind of single-parameter fractional infinitesimal transformation of Lie group and, under this transformation, give the definition and determining equation of fractional conformal invariance.

In particular, in Section 4, we present the fractional conformal invariance method and it is found that, using the new method, we can find a new kind of non-Noether conserved quantity. As a special case, we find that an autonomous fractional Birkhoffian system possesses more conserved quantities.

In Applications A–B of Sections 5–6, by using the fractional conformal invariance method, we, respectively, explore the conformal invariance and conserved quantities of a fractional Lotka biochemical oscillator and a fractional Hojman–Urrutia model.

Section 7 contains the conclusions.

2. Fractional Birkhoffian systems

The fractional Birkhoffian system is more general than the fractional Hamiltonian system. In this section, we explain briefly the fractional Birkhoffian system, and provide a fractional Birkhoffian method of constructing a family of fractional dynamical models which is an interesting result with uncertainty of a fractional problem.

Suppose that the function $f(t)$ is continuous and integrable in $[a,b]$. The Riesz–Riemann–Liouville fractional derivative is defined as [26]

$$ {}_a^R D_b^\alpha f(t) = \frac{1}{2\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^b |t-\xi|^{n-\alpha-1} f(\xi)d\xi, \quad (n\in Z^+; n-1\leqslant\alpha<n), \tag{1}$$

where n is a positive integer. By using the definition (1), we can get

$$ {}_a^R D_b^\alpha f(t) = \frac{d}{dt}({}_a^R D_b^{\alpha-1} f(t)), \quad (n-1\leqslant\alpha<n). \tag{2}$$

Let us consider the fractional Birkhoffian system of which the local coordinates of a mechanical system are determined by Birkhoff variable $a^\upsilon\ (\upsilon=1,2,\ldots,2m)$. The Birkhoffian is $B(t,\boldsymbol{a})$ and the Birkhoff functions are $R_\nu(t,\boldsymbol{a})\,(\upsilon=1,2,\ldots,2m)$, fractional derivative ${}_a^R D_b^\alpha a^\nu$ is n times continuously differentiable in $[a,b]$. If the differential equation of motion of a fractional dynamical system can be expressed in the following form [48]

$$ \begin{gathered} {}_a^R D_b^\alpha a^k = \sum_{\mu,\nu=1}^{2m} \frac{\partial {}_a^R D_b^{\alpha-1} a^k}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B(t,\boldsymbol{a})}{\partial a^\nu}, \\ \omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \omega^{\mu\nu} = \left(\left\|\omega_{\chi\delta}\right\|^{-1}\right)^{\mu\nu}, \\ (k=1,2,\ldots,2m; n-1\leqslant\alpha<n), \end{gathered} \tag{3}$$

then the system is called the fractional Birkhoffian system with Riesz–Riemann–Liouville derivative. Here $\omega_{\mu\upsilon}$ is the Birkhoff tensor and $\omega^{\mu\nu}$ is the Birkhoff contravariant tensor.

When $B, \omega^{\mu\nu}$ and ${}_a^R D_b^{\alpha-1} a^k$ do not explicitly depend on time t, Eq. (3) is reduced to the autonomous fractional Birkhoffian system.

When ${}_a^R D_b^{\alpha-1} a^k = a^\mu$, Eq. (3) is reduced to

$$ {}_a^R D_b^\alpha a^\mu = \sum_{\nu=1}^{2m} \omega^{\mu\nu}\frac{\partial B(t,\boldsymbol{a})}{\partial a^\nu}, (\mu=1,2,\ldots,2m; n-1\leqslant\alpha<n). \tag{4}$$

When ${}_a^R D_b^{\alpha-1} a^k = a^\mu$ and $0\leqslant\alpha<1$, Eq. (3) is reduced to

$$ {}_a^R D_b^\alpha a^\mu = \sum_{\nu=1}^{2m} \omega^{\mu\nu}\frac{\partial B(t,\boldsymbol{a})}{\partial a^\nu}, (\mu=1,2,\ldots,2m; 0\leqslant\alpha<1). \tag{5}$$

When $\alpha=1$, the system (3) is reduced to the integer-order Birkhoffian system

$$ \dot a^\mu = \sum_{\nu=1}^{2m} \omega^{\mu\nu}\frac{\partial B(t,\boldsymbol{a})}{\partial a^\nu}, (\mu=1,2,\ldots,2m). \tag{6}$$

In the process of establishing the differential equation of motion of an actual dynamical system, how can we embody and describe the uncertainty of a fractional and nonlinear problem? This is a very fundamental and important problem. We have explored this problem in Eq. (3). For an actual dynamical system which is given by Birkhoffian B and Birkhoff contravariant tensor $\omega^{\mu\nu}$, ${}_a^R D_b^{\alpha-1} a^k$ in Eq. (3) embodies the uncertainty of constructing a fractional dynamical model. So, in order to construct an actual fractional dynamical model, we need define

$$ \begin{gathered} {}_a^R D_b^{\alpha-1} a^k = f_k\left(t, a^\upsilon, a^{\upsilon(1)},\ldots,a^{\upsilon(n-2)}, {}_a^R D_b^{\beta-1} a^\upsilon\right), \\ \left(k,\upsilon=1,2,\ldots,2m; n-1\leqslant\alpha,\beta<n; n\in Z^+\right), \end{gathered} \tag{7}$$

here $a^{\upsilon(1)}=\frac{da^\nu}{dt},\ldots,a^{\upsilon(n-2)}=\frac{d^{(n-2)}a^\nu}{dt^{(n-2)}}$. And, if and only if

$$ {}_a^R D_b^{\alpha-1} a^k = f_k = a^k, \quad (k=1,2,\ldots,2m; n-1\leqslant\alpha<n), \tag{8}$$

Eq. (3) can give the fractional dynamical model corresponding to the integer dynamical model.

For an actual dynamical system which is given Birkhoffian B and Birkhoff contravariant tensor $\omega^{\mu\nu}$, using the fractional Birkhoffian equation (3) and choosing the condition (7) or (8), the fractional dynamical models of this system can be established.

It is worth pointing out that, for an actual dynamical system, using Eqs. (3) and (7), we can get a family of fractional dynamical models. This will be a novel and interesting result which can embody the uncertainty of a fractional problem.

3. Conformal invariance of fractional Birkhoffian systems

Now in the space $(t,\boldsymbol{a})$ of which the local coordinates are determined by Birkhoff variable $a^k\ (k=1,2,\ldots,2m)$, we give a more general fractional Lie transformation and define the fractional infinitesimal generator vector with pth extension and fractional extension. And then, under this transformation, we present the definition and determining equation of fractional conformal invariance for the fractional Birkhoffian representation.

Let us introduce a single-parameter fractional infinitesimal transformation of Lie groups with an infinitesimal parameter ε

$$ \begin{gathered} t^* = t + \varepsilon\xi(t,a^\upsilon) + O\left(\varepsilon^2\right), \\ a^{k*} = a^k + \varepsilon\eta_k(t,a^\upsilon) + O\left(\varepsilon^2\right), \quad (\upsilon,k=1,2,\ldots,2m), \end{gathered} \tag{9}$$

And the infinitesimal generators ξ and η_k are determined by

$$ \xi = \frac{\partial t^*}{\partial\varepsilon}\Big|_{\varepsilon=0}, \quad \eta_k = \frac{\partial a^{k*}}{\partial\varepsilon}\Big|_{\varepsilon=0}. \tag{10}$$

The pth extension and fractional extension of transformation (9) are, respectively

$$ \begin{gathered} \frac{d^p a^{k*}}{dt^{*p}} = \frac{d^p a^k}{dt^p} + \varepsilon\eta_k^p\left(t,a^\upsilon,a^{\upsilon(p)}\right) + O\left(\varepsilon^2\right), \\ {}_a^R D_b^\gamma a^{k*} = {}_a^R D_b^\gamma a^k + \varepsilon\eta_k^\gamma\left(t,a^\upsilon,{}_a^R D_b^\gamma a^\upsilon\right) + O\left(\varepsilon^2\right), \\ (\upsilon,k=1,2,\ldots,2m; p=1,2,\ldots; \gamma\geqslant 0). \end{gathered} \tag{11}$$

where $a^{\upsilon(1)}=\frac{da^\upsilon}{dt}, a^{\upsilon(2)}=\frac{d^2a^\upsilon}{dt^2},\ldots,a^{\upsilon(p)}=\frac{d^pa^\upsilon}{dt^p}$, and η_k^p represents the pth extended infinitesimal function, and η_k^γ represents the fractional extended infinitesimal function. And the fractional infinitesimal generator vector is defined as

$$ X^{(\gamma)} = \xi\frac{\partial}{\partial t} + \sum_{l=1}^{2m}\eta_l\frac{\partial}{\partial a^l} + \sum_{l=1}^{2m}\sum_p \eta_l^p\frac{\partial}{\partial a^{l(p)}} + \sum_{l=1}^{2m}\eta_l^\gamma\frac{\partial}{\partial {}_a^R D_b^\gamma a^l}, \quad (p=1,2,\ldots; \gamma\geqslant 0), \tag{12}$$

where η_l^p is defined by prolongation formula

$$ \eta_l^p = D\eta_l^{p-1} - a^{l(p)}D\xi, $$