International Journal of Non-Linear Mechanics 86 (2016) 112–121

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Implementation of surface effects in three kinds of finite bending

T. Sigaeva*, A. Czekanski

Department of Mechanical Engineering, Lassonde School of Engineering, York University, Toronto, Ontario, Canada

ARTICLE INFO

Article history:
Received 24 May 2016
Received in revised form
26 July 2016
Accepted 27 July 2016
Available online 28 July 2016

Keywords:
Hyperelasticity
Surface mechanics
Pure bending
neo-Hookean energy
Surface constitutive modeling

ABSTRACT

This paper demonstrates the methodology for implementing surface elasticity and surface tension in pure bending representatives of the well-known large deformation families: bending of a rectangular block, bending of one cylindrical sector into another sector of a different curvature and straightening of a cylindrical sector. Constitutive equations based on the surface energy decoupling are proposed to model the surface behavior. Changes in mechanical response of hyperelastic bodies at smaller scales, particularly surface effects on the resultant force, resultant moment and bending stiffness are analyzed for the special case of a neo-Hookean solid and found to be consistent in all three pure bending problems.

1. Introduction

Nanostructured materials, nanoscale structural elements and nanodevices have become pervasive in modern world due to their auspicious mechanical, thermal, electrical, optical and magnetic properties. Central to the discipline of nanomechanics is the observation that the overall deformation of the bulk at nano-scale is significantly affected by the bounding surface. Hence, classical continuum mechanics models should be generalized to capture the changes that happens with the body when the scale of the problem decreases and the surface-to-volume ratio becomes considerable.

An outstanding theoretical insight on the surface and interface effects in solids was firstly introduced by Gurtin and Murdoch [1,2]. The study builds on the assumption suggesting that the surface or interface can be represented by a zero-thickness membrane possessing its own mechanical properties and surface tension. As a result, the surface stresses develop around the boundary or interface of the body when it is deformed, while the residual stresses appear here as well, even in the absence of external loadings. Later, Steigmann and Ogden have generalized this model even further by replacing the membrane with a shell to account for its flexural resistance [3]. Mathematical explanation of Gurtin-Murdoch is given in the recent publication by Ru [4], while Eremeyev provides a detailed comparison with the Steigmann–Ogden model [5].

In the framework of linear elastic deformations, the above theories were employed in enormous amount of works to show the pronounced effects of surface (see works [6–8], literature reviews [9,10] and references contained therein). As for the finite deformations, studies considering the surface or interface effects in the framework of hyperelasticity appear less frequently [11–13]. This is surprising considering that nanorubbers and nanofillers are the emerging applications in rubber industry. We believe that one of the explanations for the gap between the number of available studies in the framework of linear elasticity and non-linear elasticity is the absence of surface material constants for typical elastomers. In fact, to the best of authors' knowledge, surface material constants were determined only in the framework of small deformations for a limited number of metals using atomistic simulations and highly advanced experimental measurements [14]. Another arising reason for this gap can be due to the lack of research into constitutive modeling of hyperelastic surfaces. Research on the nonlinear behavior of bulk material is extensive, while options for surface modeling are very limited. That is why one of the specific features of this work is our proposed general form of constitutive equations using decoupled energy formulation.

When the form of energy density does not matter, it is very common to analyze different material effects in the well-known families of universal solutions, which are named accordingly because they are universal to all nonlinear incompressible isotropic material and all material models (Truesdell provides the detailed description of these families [15]). A couple of studies have focused on some representatives of large deformations families taking into account the effects of surface. Altenbach et al. have demonstrated the critical effect of residual surface stresses and surface stresses on the mechanical response of a circular cylinder subjected to tension [16]. This research has previously been extended by including torsional deformations and changes in the

* Corresponding author.

http://dx.doi.org/10.1016/j.ijnonlinmec.2016.07.010
0020-7462/

Poynting effect and torsional rigidity due to the surface have been highlighted [17]. Bending of a rectangular block was tackled using Steigmann–Ogden surface model [18], but the emphasis of the work was the bifurcation analysis, not changes in the effective material properties at smaller scale. Here we would like to study three pure bending representatives of large deformations families using Gurtin–Murdoch approach, particularly problems of bending of a rectangular block, bending of one cylindrical sector into another sector of a different curvature, and straightening of a cylindrical sector. Our motivation here is not only to demonstrate the impact of surface on certain effective material properties such as bending rigidity and test the proposed constitutive equations; but also to clarify the procedure of implementation of the surface model using three mathematically different but physically similar deformations that should lead to consistent mechanical responses. In our opinion, this could trigger research interest to the area of surface hyperelasticity.

This paper is organized as follows: in Section 2 we consider the methodology for implementing surface effects and suggest the form of constitutive equations based on decoupled energy formulation, derivation details of which are summarized in the Appendix. Additionally, in the subsection of Section 2, the neo-Hookean energy density form for both bulk and surface is detailed. This is followed by the discussion on the surface material parameters used in numerical calculations. Section 3 contains three subsections devoted to the aforementioned different pure bending problems: bending of a rectangular block, bending of one cylindrical sector into another sector of a different curvature, straightening of a cylindrical sector. Each of these subsections has the description on how surface effects were implemented and contains discussions on numerical calculations and consistency of derived results. Conclusions of the conducted study are summarized in Section 4.

2. Methodology

We consider the equilibrium of a deformable solid occupying a region Ω_0 and made of a homogeneous, initially isotropic and incompressible hyperelastic material. Part of the surface $\partial\Omega_0^s$ of this body is coated with a thin reinforcing film representing the surface effects. This film is made of a homogeneous, isotropic hyperelastic material with elastic constants different from those of the bulk. The rest of the boundary is denoted by $\partial\Omega_0^b$, so that $\partial\Omega_0 = \partial\Omega_0^b \cup \partial\Omega_0^s$ (Fig. 1a). When subjected to external loadings, the deformation gradients $\boldsymbol{F}$ and $\boldsymbol{F}^s$ are used to map the body and coating, respectively, from reference configuration Ω_0 and $\partial\Omega_0$ to current configuration Ω and $\partial\Omega = \partial\Omega^b \cup \partial\Omega^s$ (Fig. 1b).

Since the bulk is made of an incompressible material, the constitutive law relating the Cauchy stress tensor $\boldsymbol{\sigma}$ with material properties looks as follows:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mathcal{W}_1\boldsymbol{B} - 2\mathcal{W}_2\boldsymbol{B}^{-1},$$

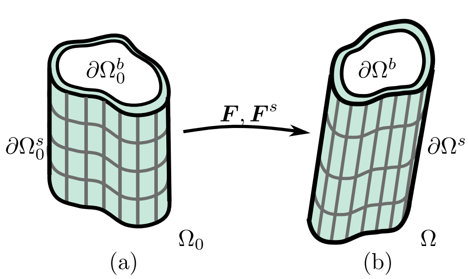

Fig. 1. Common reference configuration (a) and current configurations (b) for the body with surface effect.

Here p is the Lagrange multiplier to ensure incompressibility, $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy–Green strain tensors, $\mathcal{W}_i = \frac{\partial \mathcal{W}}{\partial I_i}$ $(i=1,2)$ with invariants $I_1 = \mathrm{tr}\boldsymbol{B}$ and $I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{B})^2 - \mathrm{tr}\boldsymbol{B}^2\right]$.

Since the surface is modeled as a thin film perfectly attached to the bulk and all deformations along the thickness are uniform, we can state that the surface area is not constant. Thus, as for regular compressible bodies [19], it makes sense to consider its surface area-changing and area-preserving deformations separately. To this end, we assume that the surface strain energy is decoupled into the part $\mathcal{U}_{const}$ that does not change the area of the surface and the part $\mathcal{U}_{var}$ that does, i.e.,

$$\mathcal{U} = \mathcal{U}_{const}(\bar{\boldsymbol{B}}^s) + \mathcal{U}_{var}(J^s) \quad \text{with } J^s = \det \boldsymbol{F}^s, \quad \boldsymbol{B}^s = \boldsymbol{F}^s\boldsymbol{F}^{sT},$$
$$\bar{\boldsymbol{F}}^s = (J^s)^{-1/2}\boldsymbol{F}^s, \quad \det\bar{\boldsymbol{F}}^s = 1, \quad \bar{\boldsymbol{B}}^s = \bar{\boldsymbol{F}}^s\bar{\boldsymbol{F}}^{sT} = (J^s)^{-1}\boldsymbol{B}^s. \tag{1}$$

Here J^s is the surface area ratio, $\boldsymbol{B}^s$ represents the left Cauchy–Green surface strain tensor, while $\bar{\boldsymbol{F}}^s$ and $\bar{\boldsymbol{B}}^s$ are the intentionally introduced (unimodular) deformation gradient and left Cauchy–Green strain tensors associated with surface area-preserving deformations and energy $\mathcal{U}_{const}$.

Using lengthy algebraic formulations and the split form of the surface energy density (1), we derive the decoupled form of constitutive law (see Appendix for details)

$$\boldsymbol{\sigma}^s = \boldsymbol{\sigma}^s_{const}(\bar{\boldsymbol{B}}^s) + \boldsymbol{\sigma}^s_{var}(J^s) = \frac{2}{J^s}\mathrm{dev}^s\left[\bar{\boldsymbol{B}}^s\frac{\partial\mathcal{U}_{const}}{\partial\bar{\boldsymbol{B}}^s}\right] + \frac{\partial\mathcal{U}_{var}}{\partial J^s}\boldsymbol{I}^s, \tag{2}$$

where $\boldsymbol{\sigma}^s$ is the surface Cauchy stresses, $\boldsymbol{I}^s$ is the second-order surface identity tensor and "dev^s" is used to denote a deviatoric part of a second-order surface tensor, which is expressed for an arbitrary surface tensor $\boldsymbol{M}$ as

$$\mathrm{dev}^s\boldsymbol{M} = \boldsymbol{M} - \tfrac{1}{2}\mathrm{tr}(\boldsymbol{M})\boldsymbol{I}^s.$$

Also, derivative of a scalar-valued tensor function $f(\boldsymbol{M})$ with respect to surface tensor $\boldsymbol{M}$ is the following second-order tensor $\frac{\partial f(\boldsymbol{M})}{\partial \boldsymbol{M}} = \frac{\partial f(\boldsymbol{M})}{\partial M_{ij}}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Here symbol "$\otimes$" is used to denote a usual tensor product or dyad of the orthonormal basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$.

Let us look at the following arguments to explain what premises for the decoupled constitutive law (2) are. First of all, the decoupled volumetric and isochoric formulation itself is commonly used for constitutive models of compressible hyperelastic materials [19,20]. The term $\boldsymbol{\sigma}^s_{var}(J^s)$, denoting the surface area changing stresses, has been well established in surface mechanics to represent surface tension or residual effect appearing at smaller scales [1,5]. For better understanding of the surface area preserving stresses $\boldsymbol{\sigma}^s_{const}(\bar{\boldsymbol{B}}^s)$ and particularly its dependency on $\bar{\boldsymbol{B}}^s$, it first should be noted that many constitutive models such as neo-Hookean and Arruda-Boyce [21] have physical interpretations which depend on a volumetric density of polymer chains. As such when used for a compressible material these theories make the most sense when they depend on the unimodular part of the left Cauchy–Green deformation tensor, in terms of which the isochoric stresses are formulated. Although, significantly less theoretical work has been completed for the elasticity of surfaces which undergo large deformations and we do not know of any micromechanical models which provide a physical interpretation of surface elasticity, we postulate that, like in the case compressible hyperelasticity, the large changes in surface area induced by the finite deformations will significantly alter the aerial density of polymer chains on the surface of the material. Hence we believe that there is a stronger physical motivation to express the contribution to the surface stress from surface elasticity in terms of $\bar{\boldsymbol{B}}^s$ than on $\boldsymbol{B}^s$ as was done in the past [16].

Finally, to couple deformations of the solid and thin film, we introduce the following boundary value problem, for which addition of the surface effects gives a rise to a nonstandard boundary