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The mechanics of clearance in a non-Newtonian lubrication layer

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ABSTRACT

This paper investigates the mechanics of clearance of an embedded particle in a lubrication layer of viscoelastic fluid. We show theoretically that in a slider bearing domain containing a viscoelastic fluid, the oscillating shearing motion of a wall aids in transporting away any embedded particle towards the moving boundary. The impact of geometry and material properties of the fluid layer are explored by coupling theoretical and numerical methods. Our approach suggests a possible mechanism by which the human eye could clear out any debris beneath the eyelid, under responsive blinking. Our simplified analysis brings to bear interesting approaches from physics and engineering upon a very complex biological problem and could provide essential clues about the physiological design of the tear film.

1. Introduction

In this paper we discuss a fluid solid interaction problem. Specifically, we study the motion of a particle in a lubrication layer of a viscoelastic fluid which is driven by the quasi-periodic shearing motion of its boundary. The current paper is an extension of the work by Huang et al. [17] who discuss the lubrication flow of second grade fluids with constant viscosity in a similar geometry to our own, but goes beyond it as well. The literature on lubrication flows of non-Newtonian fluids is vast. Some studies of marginal relevance to our own include those by Park and Kwon [31] who obtained numerical solution for the non-inertial lubrication flow for power law fluids and Bujurke et al. [8], who examine lubrication flow and load carry capacity of a second order fluid in a geometry with approaching parallel surfaces. In the papers by Bourgin [4], Bourgin and Tichy [5] and Sawyer and Tichy [35] a perturbation method in the Deborah number, *De*, is used to obtain an approximate solution to the viscoelastic lubrication equations with a second order fluid and with various boundary conditions. More recently, Shah et al. [38] used the homotopy method to obtain approximate solutions to the lubrication flow in a slider bearing geometry with a power-law fluid. In all of the above papers the viscoelastic parameter, the Deborah number *De* was seen to strongly effect the flow properties as well as the shear-dependent viscosity index. Our paper is a generalization of these previous studies. Also, this work goes beyond the examination of the flow; we additionally consider the induced motion of an inertial particle trapped in this fluid layer.

The problem is studied in two parts: (i) the flow of a visocoelastic fluid in a slider bearing domain using analytic techniques and (ii) the induced motion of any tracer particle embedded in the fluid layer (see Fig. 1) due to oscillatory shearing motion of one of the boundaries, studied numerically. *Our initial hypothesis is that the fluid layer, being non-Newtonian and viscoelastic, serves the purpose of transporting out any embedded particles, away from the stationary boundary.* Once we have a way to compute the internal forces of the fluid, we can model their influence on foreign bodies in the fluid using the model suggested by Wiberg and Smith [43]. In their paper, the authors suggest an empirical model which accounts for drag and lift forces in addition to added mass and the Basset forces induced upon a suspended particle in a fluid.

This problem is motivated by the fundamental mechanics of the tear film flow in the eye. Our analysis helps shed light on the mechanics of tear film under blinking motion of the eyelid and the process of debris clearance. This study is also pertinent to any area which is concerned with lubrication and protection. The current study builds on the existing studies concerned with viscometric flows of non-Newtonian fluids [23,11,15,17,34]. In the eye, the fluid that serves as the lubricating layer (see Fig. 1) is thought to consist of a very thin lipid layer, an aqueous layer, and a mucus layer although modern theories suggest a less distinct parts demarcation of the tear film [10,13,18,30]. In any case, the bulk of the tear film is composed of what most people think of when they think of tears: the aqueous layer [27]. This layer, which is enriched saline, serves to moisten the eye and provide nutrients and is present both on the surface of the open eye and under the eyelid [28]. Mucus is a secreted fluid that is a sticky water-insoluble gel

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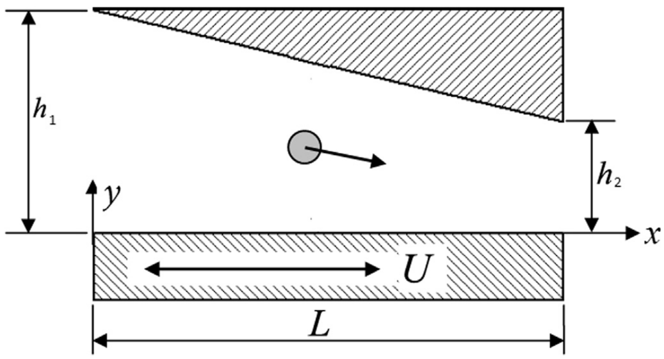

Fig. 1. A schematic of the slider bearing geometry employed in this study. The top plate is held fixed while the bottom plate slides with velocity U. The slope of the top plate is varied based on appropriate choice of h_1 and h_2.

formed by non-covalent linkages. The changes over time of the linkages within the mucus generate the flow properties of this gel. The mucus material clings to epithelial surfaces serving the primary purposes of protection and lubrication [1,12,42]. The lipid layer serves mostly to reduce evaporation of the aqueous layer and resides mostly between the two edges of the eyelid and will be ignored in this study [7]. The aqueous layer behaves strictly as a Newtonian fluid [16] while the mucus layer, which lies between the aqueous layer and the surface of the eyeball displays non-Newtonian characteristics; its molecular composition gives it a shear dependent and elastic character [29]. In addition to serving as a lubricant for the eyelid and as an adhesive that keeps the aqueous layer coating of the eyeball in place, it also serves in the role of a protector of the eye. Also, along with certain molecules and enzymes that work to chemically preventing disease from reaching the eye, there is some unclarified process whereby mucin moleculars wrap around unwelcome particles and serve to remove them from the tear film [20]. Details about the physiological properties of the tear film can be found in the literature [9,13,18,30,41,45]. It is important to note that the analysis in this paper takes up a toy model to understand the mechanics of protection and also to clarify the role of viscoelasticity in the overall process of clearance; the parameters chosen in this study do not coincide with those relevant to the eye. We propose a fluid model with just a single homogeneous fluid layer given by the second-order non-Newtonian fluid equations with variable viscosity, referred to as the *modified second order fluid* equations [21,22,24]. It is clear that the overall system is a very complex one where much more remains to be understood, about physical domain and material makeup of the tear film. A full fledged modeling of this system remains a daunting task and we see it best to approach the problem in a series of less complicated steps.

In Section 2, we treat the problem of steady flow of the modified second order fluid in a slider bearing geometry. Asymptotic analytical solutions for the fluid flow and stress induced upon the walls are first obtained. Section 3 then treats the formulation of the particle equations coupled to a quasi-steady version of the flow derived earlier and its numerical solutions. The paper concludes with a discussion of our results and the impact of parameters such as the shear-rate index, magnitude of viscoelasticity and slope of the channel walls, in Section 4.

2. The fluid equations

We consider the flow of a viscoelastic fluid with variable viscosity in a slider bearing geometry (see Fig. 1). The lower plate is capable of moving with a constant velocity, $\mathbf{U}$, while the upper plate is fixed at a fixed angle, a thus allowing for deviations from a perfectly parallel plate situation.[1] The equation $h(x) = h_1 + mx$, with $h(0) = h_1$ and $h(L) = h_2$, describes the distance between the two plates as a function of x and m is the slope of the top plate; in principle, h can be any function of x. The governing equations describing the system are given by the conservation of linear momentum:

$$div\ \mathbf{T} = \frac{D\mathbf{u}}{Dt} \tag{1}$$

where $\mathbf{u} = (u, v, w)$ indicates the velocity field, $\frac{D}{Dt}$ represents the total derivative and the stress tensor is given by

$$\mathbf{T} = -\,p\ \mathbf{I} + \eta\ \mathbf{A}_1 + \alpha_1\mathbf{A}_1^2 + \alpha_2\mathbf{A}_2 \tag{2}$$

$$\eta\left(\dot{\gamma}\right) = \eta_0\,|1\,+\kappa\,\dot{\gamma}|^q \tag{3}$$

representing a generalized second grade fluid [24,21,22]. Here p is the isotropic pressure and α_1, α_2 are the first and second normal stress coefficients, respectively. The viscosity is taken to be of the power-law type in order to capture shear-thinning and thickening behavior for varying values of q, with $\dot{\gamma} = \frac{1}{2}\sqrt{|\mathbf{A}_1 \colon \mathbf{A}_1|}$ representing the shear rate. We define the deformation tensors in Eq. (21) as

$$\mathbf{A}_1 = \nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^{\mathrm{T}} \tag{4}$$

$$\mathbf{A}_2 = \mathbf{u}\nabla(\mathbf{A}_1) + \mathbf{A}_1\ \nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^{\mathrm{T}}\mathbf{A}_1. \tag{5}$$

Additionally, the incompressibility condition applies, hence

$$\mathrm{div}\left(\mathbf{u}\right) = 0 \Rightarrow \partial_x u + \partial_y v = 0. \tag{6}$$

Since the problem is independent of the z-direction, we treat the problem as two dimensional. We use Eq. (6) and scale approximations, $v \gg u$, $\frac{\partial u}{\partial x} \gg \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x} \gg \frac{\partial v}{\partial y}$, and assume time independence, i.e. steady state flow, to simplify the above equations. The scale approximations and incompressibility suggest that (i) $\left|\frac{v}{u}\right| = \epsilon$ and (ii) $\left|\frac{h}{L}\right| = \epsilon$. As a result, the leading order components, up to $o(\epsilon)$, of the extra stress tensor reduce to the form:

$$\tau_{xx} = -\,2\eta\left(\dot{\gamma}\right)\left(\partial_x u\right) + \alpha_1\left(\partial_y u\right)^2 + 2\alpha_1\partial_x u\partial_y u \tag{7}$$

$$\tau_{yy} = -\,2\eta\left(\dot{\gamma}\right)\left(\partial_y v\right) + \alpha_1\left(\partial_y u\right)^2 + 2\alpha_2\left[\left(\partial_y u\right)^2\right] \tag{8}$$

$$\tau_{xy} = -\,2\eta\left(\dot{\gamma}\right)\left(\partial_y u\right) + 2\alpha_2\left[2u\partial_x\partial_y u + v\partial_y\partial_y u + 2\partial_x u\partial_y u\right] \tag{9}$$

For arbitrary values of $q \in \mathbf{R}$ the problem remains analytically unsolvable, except in simple cases such as $q = 1$ or 2. However, the problem becomes tractable in the special case of $|q| \ll 1$ leading us to approximate the viscosity function using the binomial approximation:

$$\eta\left(\dot{\gamma}\right) = \eta_0\left(1\,+\kappa\dot{\gamma}\right)^q \approx \eta_0(1\,+q\kappa\dot{\gamma}/2).$$

As a result, the components of the extra stress tensor, with the lubrication approximation applied to the viscosity term, reduce to the form

[1] We model the problem being studied, in the standard manner as one where the bottom plate is moving while the top plate is fixed. This is the reverse case to that of the human eye where the eyelid is akin to the bottom moving plate. Since gravity is being ignored in this problem, our choice can be justified by frame invariance.