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Symmetries of the hyperbolic shallow water equations and the Green–Naghdi model in Lagrangian coordinates

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The observation that the hyperbolic shallow water equations and the Green–Naghdi equations in Lagrangian coordinates have the form of an Euler–Lagrange equation with a natural Lagrangian allows us to apply Noether's theorem for constructing conservation laws for these equations. In this study the complete group analysis of these equations is given: admitted Lie groups of point and contact transformations, classification of the point symmetries and all invariant solutions are studied. For the hyperbolic shallow water equations new conservation laws which have no analog in Eulerian coordinates are obtained. Using Noether's theorem a new conservation law of the Green–Naghdi equations is found. The dependence of solutions on the parameter is illustrated by self-similar solutions which are invariant solutions of both models.

1. Introduction

There are two distinct ways to model phenomena in continuum mechanics. The typical approach uses Eulerian coordinates, where flow quantities at each instant of time during motion are described at fixed points. In this reference frame, one studies individual spatial positions, regardless of what particles reach those positions at a given instant of time. Alternatively, the Lagrangian description is used, where the particles are identified by the positions which they occupy at some initial time. Typically, Lagrangian coordinates are not applied in the description of fluid motion. One reason is that experimental data are simply presented in Eulerian coordinates. A second reason is that the coordinate system in the Lagrangian description is moving: it can be twisted and distorted. Thirdly, the Lagrangian type of specification leads to a cumbersome analysis [1]. However, in some special contexts the Lagrangian description is indeed useful in solving certain problems. This paper gives one more of such demonstrations.

1.1. Equations of fluids with internal inertia

In the present paper we consider that the dispersive shallow water Green–Naghdi model in Lagrangian description is studied.

The Green–Naghdi [2] equations, which are also known as the Serre [3] or Su–Gardner [4] equations, model the fully nonlinear and weakly dispersive surface gravity waves on fluid of finite depth. Solutions of the Green–Naghdi equations approximate irrotational flows described by the Euler equations. A large number of works have been devoted to the studies of the Green–Naghdi equations from both analytical and numerical points of view.[1]

One of the representations of the Green–Naghdi model can be considered as a particular class of the equations [7,8]

$$\dot\rho + \rho\,\mathrm{div}\left(u\right) = 0, \quad \rho\dot u + \nabla p = 0,$$

$$p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\dot\rho}\right) - \mathrm{div}\left(\frac{\partial W}{\partial\dot\rho}u\right)\right) - W, \tag{1}$$

where $W(\rho, \dot\rho)$ is a given potential, "dot" denotes the material time

[1] See, for instance, [5,6], and references therein.

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derivative: $\dot{f} = \frac{df}{dt} = f_t + u\nabla f$ and $\frac{\delta W}{\delta \rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u.

The models (1) were derived [7,8] using the Lagrangian

$$\mathcal{L} = \rho\frac{u^2}{2} - W\left(\rho, \dot{\rho}\right). \tag{2}$$

The Green–Naghdi model corresponds to the potential $W = \gamma_1\rho^2 - \gamma\rho\dot{\rho}^2$. In particular, the potential for classical hyperbolic shallow water equations is determined by the condition $\gamma = 0$. The class of dispersive models (1) is an example of a medium whose behavior depends not only on the thermodynamical variables but also on their derivatives with respect to space and time. In this particular case the potential function depends on the total derivative of the density, which reflects the dependence of the medium on its inertia.

1.2. Symmetries and conservation laws

Symmetries have always attracted the attention of scientists. One of the tools for studying symmetries is the group analysis method [9–13], which is a basic method for constructing exact solutions of partial differential equations. The group properties of the Green–Naghdi model were studied in [14]. The hyperbolic shallow water equations $(\gamma = 0)$ were considered in [15–17]. Invariant and partially invariant solutions of these equations were analyzed there. Notice also that the complete group classification with respect to the potential $W(\rho, \dot{\rho})$ of Eqs. (1) was performed in [18,19].

Besides assisting with the construction of exact solutions, the knowledge of an admitted Lie group allows one to derive conservation laws. Conservation laws provide information on the basic properties of solutions of differential equations, and they are also needed, in the analyses of stability and global behavior of solutions. Noether's theorem [20] is the tool which relates symmetries and conservation laws. However, an application of Noether's theorem depends on the following condition: the differential equations under consideration can be rewritten as Euler–Lagrange equations with appropriate Lagrangian. Among approaches which try to overcome this limitation one can mention here the approaches developed in [21–24].[2]

Recently the authors observed that, in Lagrangian coordinates, Eqs. (1) are equivalent to the Euler–Lagrange equation with a natural Lagrangian.[3] This allows us to use Noether's theorem directly, without any additional constructions.

1.3. Eulerian and Lagrangian coordinates

Relations between Lagrangian coordinates (t, x_0) and Eulerian coordinates (t,x) are defined by the condition $x = \varphi(t, x_0)$, where the function $\varphi(t, x_0)$ is the solution of the Cauchy problem

$$\varphi_t(t, x_0) = u(t, \varphi(t, x_0)), \quad \varphi(t_0, x_0) = x_0.$$

By virtue of the definition of Lagrangian coordinates, one can assume that $\varphi_{x_0} > 0$.

In Lagrangian coordinates, the general solution of the mass conservation law equation is

$$\rho(t, \varphi(t, x_0)) = \frac{\rho_0(x_0)}{\varphi_{x_0}(t, x_0)},$$

where $\rho_0(x_0)$ is an arbitrary function of the integration. Without loss of generality one can assume that $\rho_0 = 1$. In fact, using the change

$$\xi = \alpha(x_0), \tag{3}$$

where $\alpha'(x_0) = \rho_0(x_0)$, one obtains that $\bar{\rho}(t, \xi) = \frac{1}{\bar{\varphi}_\xi(t, \xi)}$. Here the functions $\bar{\varphi}(t, \xi)$ and $\varphi(t, x_0)$ are related by the formula[4]

$$\bar{\varphi}(t, \alpha(x_0)) = \varphi(t, x_0).$$

In the Lagrangian coordinates the change (3) defines an equivalence transformation: this change simplifies the equations studied. Because the variable ξ is related with $\rho_0(x_0)$, it is called the Lagrangian mass coordinate [26].

1.4. Objectives of the present paper

In the present paper we study the equations

$$\rho_t + u\rho_x + \rho u_x = 0,$$
$$\rho(u_t + uu_x + 2\gamma_1\rho_x) = 2\gamma\left(\rho^3(u_{xt} + uu_{xx} - u_x^2))\right)_x,$$

where ρ is the water depth, u is the horizontal velocity, g is the gravity, and ε is the ratio of the vertical length scale to the horizontal length scale. Here for the sake of convenience we use

$$\gamma_1 = \frac{g}{2}, \quad \gamma = \frac{\varepsilon^2}{6}.$$

The study depends on γ: $\gamma = 0$ gives the classical hyperbolic shallow water equations, while $\gamma > 0$ gives the Green–Naghdi equations.

As the transition to the Lagrangian coordinates is not a point transformation, group analysis of these equations has to be performed independently of their representations in Eulerian coordinates. In particular, group classification of the admitted Lie algebras and invariant solutions of the shallow water equations ($\gamma = 0$ and $\gamma \neq 0$) in Lagrangian coordinates are obtained in the present paper. Noting that the equations in Lagrangian coordinates are equivalent to the Euler–Lagrange equation

$$\frac{\delta \mathcal{L}}{\delta \varphi} = 0, \tag{4}$$

Noether's theorem is applied to them, where $\frac{\delta}{\delta\varphi}$ is the variational derivative, and $\mathcal{L} = \mathcal{L}(\xi, \varphi, \varphi_\xi, \varphi_{t\xi})$.

The paper is organized as follows. In Section 2 the necessary background for applying symmetries to derive conservation laws is given. The hyperbolic shallow water equations in Lagrangian coordinates are analyzed in Section 3, where invariant solutions and conservation laws are derived. In Section 4 the Green–Naghdi equations are considered, and comparisons of invariant solutions and conservation laws in Lagrangian and Eulerian variables are given. Using self-similar solutions, the influence of the parameter γ is presented, while a new conservation law of the Green–Naghdi equations is also found.

2. Noether's theorem

We begin with the background related to the application of symmetries for constructing conservation laws.[5]

Let

[2] Therein one can find more details and references.

[3] After the manuscript was completed we discovered the paper [25] which led us to a series of references, where the magnetohydrodynamics and gas dynamics equations have been studied in Lagrangian coordinates considering them as Euler–Lagrange equations.

[4] The sign ¯ is further omitted.

[5] The reader is referred to [27] for the details on symmetries and conservation laws.