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On the bounded symmetrical motions in the three-body problem

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ABSTRACT

We study a special case of the three-body problem where two bodies are the same mass and there is a manifold of symmetrical motions. We find conditions of existence of bounded symmetric motions. To analyze stability, we substantially rely on the structure of the manifold of symmetric motions and use integrals of energy and angular momentum.

1. Introduction

When considering symmetrical motions in the three-body problem, it is usual to refer to Sitnikov [1] who managed to prove the existence of oscillating final evolutions in the case where two bodies have equal mass. These are evolutions that exhibit symmetry. Although Sitnikov's statement was in truth proved rigorously only for the restricted three-body problem, nevertheless it was an impetus to further research in this direction, and eventually, it found its justification for the general three-body problem under certain additional restrictions [2–4]. Note that the existence of oscillating final evolutions in the three-body problem was postulated by Chazy [5].

In our paper, we consider an "inverse" problem by exploring conditions under which bounded symmetric motions exist.

As is shown in [6], the equations of motion for the three-body problem (or three mass points) can be represented in the following form:

$$\begin{aligned}
\boldsymbol{\rho}_1'' &= \mu_2\frac{\boldsymbol{\rho}_2-\boldsymbol{\rho}_1}{|\boldsymbol{\rho}_{12}|^3}+\mu_3\frac{\boldsymbol{\rho}_3-\boldsymbol{\rho}_1}{|\boldsymbol{\rho}_{13}|^3},\\
\boldsymbol{\rho}_2'' &= -\mu_1\frac{\boldsymbol{\rho}_2-\boldsymbol{\rho}_1}{|\boldsymbol{\rho}_{12}|^3}+\mu_3\frac{\boldsymbol{\rho}_3-\boldsymbol{\rho}_2}{|\boldsymbol{\rho}_{23}|^3},\\
\boldsymbol{\rho}_3'' &= -\mu_1\frac{\boldsymbol{\rho}_3-\boldsymbol{\rho}_1}{|\boldsymbol{\rho}_{13}|^3}-\mu_2\frac{\boldsymbol{\rho}_3-\boldsymbol{\rho}_2}{|\boldsymbol{\rho}_{23}|^3},
\end{aligned} \tag{1.1}$$

where the prime sign denotes the operation of differentiation with respect to τ $(\tau = t\sqrt{GM}/r_0^{3/2})$, $\mu_i = m_i/M$, $M = m_1+m_2+m_3$, and r_0 is a parameter with the unit-length dimension. In Eq. (1.1), $\boldsymbol{\rho}_i = \mathbf{r}_i/r_0$, where $\mathbf{r}_i$ are radius vectors of points in the inertial reference system with the origin at the center of mass m_i. The variables $\boldsymbol{\rho}_i = \mathbf{r}_i/r_0$ are dimensionless since r_0 has the dimension of the length unit. In what follows, by considering the equations of motion in the form (1.1), we obtain a possibility to use dimensionless quantities; this is convenient for the subsequent transformations of the system.

In the sequel, we essentially use the fact that system (1.1) is conservative, i.e., the integral of energy

$$\frac{1}{2}\sum_i^3 \mu_i\boldsymbol{\rho}_i'^2 - \sum_{i<j}\frac{\mu_i\mu_j}{|\boldsymbol{\rho}_{ij}|} = h = \text{const} \tag{1.2}$$

does exist. We also use the vector integral of angular momentum

$$\sum_i^3 \mu_i(\boldsymbol{\rho}_i \times \boldsymbol{\rho}_i') = \mathbf{C}. \tag{1.3}$$

In addition, we assume that $\mathbf{C} \neq \mathbf{0}$.

Subtracting the first equation of (1.1) from the second one and taking into account the equality

$$\sum_i^3 \mu_i\boldsymbol{\rho}_i = \mathbf{0}$$

that corresponds to the case where the origin of the reference system coincides with the center of mass of material points under consideration, provided that $\mu_1 = \mu_2 = \mu$, we obtain the manifold of symmetric motions

$$\boldsymbol{\rho}_{12}'' = -2\mu\frac{\boldsymbol{\rho}_{12}}{|\boldsymbol{\rho}_{12}|^3} - \mu_3\frac{\boldsymbol{\rho}_{12}}{|\boldsymbol{\rho}_{13}|^3},$$

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$$\rho_3'' = -\frac{\rho_3}{|\rho_{13}|^3}, \tag{1.4}$$

on which we have $|\rho_{13}| = |\rho_{23}|$ and the equality

$$\rho_3^2 = \mu^2(-\rho_{12}^2 + 4\rho_{13}^2) \tag{1.5}$$

is true. The latter is a special case of the following general equality

$$\rho_3^2 = -\mu_1\mu_2\rho_{12}^2 + \mu_1(\mu_1+\mu_2)\rho_{13}^2 + \mu_2(\mu_1+\mu_2)\rho_{23}^2 \tag{1.6}$$

derived in [8], where we found relations that reveal dependencies between ρ_i^2 and ρ_{ij}^2 ($i, j = 1, 2, 3$), and also $\rho_i'^2$ and $\rho_{ij}'^2$, each taken separately. It is easy to see that equality (1.5) can be obtained from (1.6) by setting $\mu_1 = \mu_2 = \mu$.

The manifold of symmetric motions is characterized by the following equations:

$$\rho_{12} \times \rho_{12}' = \mathbf{C_1}, \quad \rho_3 \times \rho_3' = \mathbf{C_2}, \tag{1.7}$$

where $\mathbf{C_1}$ and $\mathbf{C_2}$ are constant vectors. This allows us to reduce the system (1.4) qualitatively to a system with two degrees of freedom. Indeed, multiplying Eq. (1.4) by $2\rho_{12}$ and $2\rho_3$ respectively, we obtain

$$\rho_{12}^2{}'' = 2v_{12}^2 - \frac{4\mu}{\rho_{12}} - 2\mu_3\frac{\rho_{12}^2}{\rho_{13}^3},$$
$$\rho_3^2{}'' = 2v_3^2 - 2\frac{\rho_3^2}{\rho_{13}^3}, \tag{1.8}$$

where $v_{12}^2 = \rho_{12}'^2$, $v_3^2 = \rho_3'^2$. By observing that

$$\rho_{12}'^2 = \rho_{12}'^2 + \frac{|\rho_{12} \times \rho_{12}'|^2}{\rho_{12}^2}, \tag{1.9}$$

$$\rho_3'^2 = \rho_3'^2 + \frac{|\rho_3 \times \rho_3'|^2}{\rho_3^2}, \tag{1.10}$$

and using (1.7), we can represent Eq. (1.8) as follows:

$$\rho_{12}^2{}'' = 2\left(\frac{(\rho_{12}^{2\prime})^2}{4\rho_{12}^2} + \frac{|\mathbf{C}_1|^2}{\rho_{12}^2}\right) - \frac{4\mu}{\rho_{12}} - 2\mu_3\frac{\rho_{12}^2}{\rho_{13}^3},$$
$$\rho_3^2{}'' = 2\left(\frac{(\rho_3^{2\prime})^2}{4\rho_3^2} + \frac{|\mathbf{C}_2|^2}{\rho_3^2}\right) - 2\frac{\rho_3^2}{\rho_{13}^3}. \tag{1.11}$$

If $\mathbf{C}_2 = \mathbf{0}$, then system (1.11) admits a motion, for which the mass point μ_3 oscillates along the axis that passes through the center of mass of the system and is orthogonal to the plane of motion of other two mass points with equal masses. Namely for this case, Sitnikov has succeeded to reveal the existence of final evolutions [1] that oscillate around the stationary motion

$$\rho_3 = 0, \quad \rho_{12} = \frac{|\mathbf{C}_1|^2}{2(\mu + 4\mu_3)}.$$

Definition 1. Following [7], we say that the pair of mass points (μ, μ) of system (1.4) is *Hill stable* if the following inequality is satisfied:

$$|\rho_{12}(\tau)| < c_1 \quad \forall \tau \in R, \ 0 < c_1 = \text{const.} \tag{1.12}$$

Definition 2. We say that a motion $\rho(\tau) = (\rho_1, \rho_2, \rho_3)^T$ of system (1.4) is *distal* if the following inequality is satisfied:

$$|\rho_{ij}(\tau)| \geq c_2 \quad \forall \tau \in R, \quad \forall i < j, \ 0 < c_2 = \text{const.} \tag{1.13}$$

Assertion. *Let $\rho(\tau) = (\rho_1, \rho_2, \rho_3)^T$ be a symmetric motion of system (1.4), which belongs to the set*

$$\Omega = \{(\rho, \rho') : T - U = h < 0\}.$$

Then, in the case where $|\mathbf{C_1}| \neq 0$, the motion is distal and the pair of points (μ, μ) is Hill stable.

Proof. Since $|\mathbf{C_1}| \neq 0$, $|\rho_{12}|$ satisfies an inequality of the form (1.13). According to (1.5), given that its left-hand side is nonnegative, we obtain

$$-\rho_{12}^2 + 4\rho_{13}^2 \geq 0, \tag{1.14}$$

that allows us to conclude that the motion is distal.

Due to inequality (1.14), we have $|\rho_{13}| \geq |\rho_{12}|/2$. Therefore, if we assume that the pair of mass points (μ, μ) is not Hill stable, then all three mutual distances $|\rho_{ij}| (i, j = 1, 2, 3)$ can be arbitrarily large. However, given that the relevant symmetric motion belongs to the set Ω, we obtain a contradiction, since at least one of the mutual distances $|\rho_{ij}|$ is always bounded on Ω at any time. Hence, we conclude that the assertion is true.

The manifold of symmetric motions has the property that by choosing initial conditions we can avoid not only triple collisions of mass points, but also double ones. So, as a result, we can achieve the distality. Unfortunately, it is not true in the general case, and the problem of double collisions remains open. □

2. A theorem on boundedness of symmetric motions

In our study of boundedness for symmetric motions, we represent the kinetic energy T both in the well-known form of expression

$$T = \frac{1}{2}(\mu_1\mu_2\rho_{12}'^2 + \mu_1\mu_3\rho_{13}'^2 + \mu_2\mu_3\rho_{23}'^2), \tag{2.1}$$

and also in the form of the following expression [9]:

$$T = \frac{1}{2}\left(\frac{\mu_3}{\mu_1+\mu_2}\rho_3'^2 + \frac{\mu_1\mu_2}{\mu_1+\mu_2}\rho_{12}'^2\right). \tag{2.2}$$

Equality (2.2) and its analogues, which involve vectors ρ_2', ρ_{13}' and ρ_1', ρ_{23}' respectively, are obtained by the author [9]. These equalities are expressions of kinetic energy represented in modified Jacobi coordinates. The modification in its essence consists in considering a pair (μ_i, μ_j) and the corresponding vector ρ_k $(k \neq i, j)$, which outgoes from the center of mass of the system in the direction of the mass μ_k $(k = 1, 2, 3)$, instead of the vector $\mathbf{R}$, which outgoes from the center of mass of a fixed pair (μ_1, μ_2) in the direction of the mass μ_3. Depending on the mass distribution in the three-body problem and/or depending on which one of the pairs (μ_i, μ_j) is Hill stable, we can use any one of the completely equivalent expressions of kinetic energy in order to obtain an appropriate Lagrangian.

In the manifold of symmetric motions (1.4), these expressions can be represented, respectively, as follows:

$$T = \frac{1}{2}(\mu^2\rho_{12}'^2 + 2\mu\mu_3\rho_{13}'^2), \tag{2.3}$$

$$T = \frac{1}{2}\left(\frac{\mu_3}{2\mu}\rho_3'^2 + \frac{\mu^2}{2\mu}\rho_{12}'^2\right), \tag{2.4}$$

and this allows us to obtain the following representations for the energy integral:

$$(\mu^2\rho_{12}'^2 + 2\mu\mu_3\rho_{13}'^2) - \frac{2\mu^2}{\rho_{12}} - \frac{4\mu\mu_3}{\rho_{13}} = 2h, \tag{2.5}$$

$$\left(\frac{\mu_3}{2\mu}\rho_3'^2 + \frac{\mu^2}{2\mu}\rho_{12}'^2\right) - \frac{2\mu^2}{\rho_{12}} - \frac{4\mu\mu_3}{\rho_{13}} = 2h. \tag{2.6}$$

By introducing notations

$$E_{12} = \rho_{12}'^2 - \frac{2}{\rho_{12}}, \quad E_{13} = \rho_{13}'^2 - \frac{2}{\rho_{13}}, \tag{2.7}$$

where E_{12} and E_{13} denote the energy of pairs (μ, μ) and (μ, μ_3) respectively, we can represent the energy integral (2.5) on the manifold of symmetric motions as follows: