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Nonlinear bending of functionally graded tapered beams subjected to thermal and mechanical loading

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ABSTRACT

Non-linear bending analysis of tapered functionally graded (FG) beam subjected to thermal and mechanical load with general boundary condition is studied. The governing equations are derived and a discussion is made about the possibility of obtaining analytical solution. In the case of no axial force along the beam, a closed form solution is presented for the problem. For the general case with axial force, the Galerkin technique is employed to overcome the shortcoming of the analytical solution. Moreover, the Generalized Differential Quadrature (GDQ) method is also implemented to discretize and solve the governing equations in the general form and validate the results obtained from two other methods. The effects of various thermal and mechanical loading on the nonlinear bending of tapered FG beam are investigated by implementing different analytical and numerical approaches.

1. Introduction

Developments in materials engineering resulted in a new type of materials with smooth and continuous variation of thermo-mechanical properties which are known as functionally graded materials (FGMs). FGMs possess various advantages over conventional composite laminates, such as reduction in thermal residual stresses and stress concentrations which may enhance effectiveness of the materials. FGMs are expected to be used in the design of many engineering structures such as dental and orthopedic implants, plasma facing bio materials, sensors, lightweight armor materials with high ballistic efficiency [1,2].

Furthermore, due to huge application of beams in different engineering fields, it is necessary to study their static and dynamic behavior both at small and large amplitudes where the latter is governed by nonlinear equations, see [3–6]. Also, recently, many researches were carried out to study nonlinear behavior of FG beams with constant cross section, see for instance [7–12]. Kang and Li [13,14] analyzed large deflection behavior of cantilever functionally graded beam subjected to an end force and end moment using numerical methods. Almeida et al. [15] applied tailored Lagrangian formulation for geometric nonlinear analysis of functionally graded beams with constant cross section.

On the other hand, studies related to the nonlinear analysis of variable cross section FG beams are limited to a few works. Shahba et al. [16] studied free vibration and stability of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions using finite element analysis. Rajasekaran [17] investigated the free vibration of axially functionally graded tapered Timoshenko beam which is centrifugally stiffened using the differential quadrature element method. A more general study on free vibration of axially functionally graded beams with non-uniform cross-section was carried out by Hein and Feklistova [18]. They used Haar wavelets to calculate natural frequencies. Rajasekaran [19] studied the buckling behavior of non-uniform functionally graded beam implementing differential transformation based dynamic stiffness approach. Large deflection analysis of tapered functionally graded beams was done by Davoodinik and Rahimi [20] in which a semi-analytical solution was presented for a particular case of cantilever beam with no axial force and a concentrated load at the end. Nguyen [21] and Nguyen and Gan [22] analyzed the large deflection behavior of tapered functionally graded beam with different types of inhomegenities by applying finite element analysis which is again restricted to cantilever beam.

It is also worthy to mention that analytical solutions are always preferable to the numerical approaches as the effects of various physical parameters can be studied in an easier way by analytical solutions. Moreover, presenting closed form solutions for non-linear differential equations with variable coefficients would be a difficult task.

In this paper, an analytical solution is presented for nonlinear bending analysis of tapered functionally graded beams with arbitrary boundary conditions subjected to thermal and mechanical loading.

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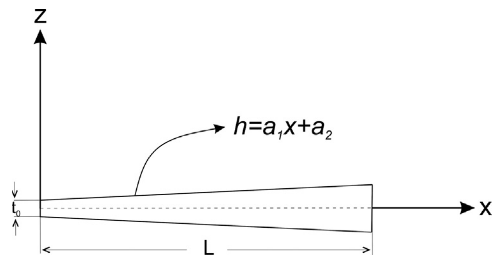

Fig. 1. Tapered beam in the Cartesian coordinate system.

The presented closed form solution is related to the case of beams with no axial force which appears when one of the edges is axially movable. Moreover, using Galerkin technique, a general solution is also obtained for the case of non-movable edges and the results are validated through comparison with Generalized Differential Quadrature (GDQ) technique.

2. Governing equations

Consider a tapered FG beam with rectangular cross section defined in Cartesian coordinate with length L in the x direction, width b in the y direction and variable thickness $2h(x)$ in the z direction as shown in Fig. 1. Linear variation is considered for the thickness of the beam as

$$h=a_1x+a_2 \tag{1}$$

where a_1 and a_2 are the taper ratio and half of the base thickness ($t_0=2a_2$), respectively. It is worthy to mention that the value of a_1 must be in a reasonable range, i.e. $0<a_1<\tan 6^\circ$, in order to achieve consistency between geometry of the beam and modeling theory. The beam is subjected to a transverse uniform load q and a thermal load caused by a uniform temperature gradient ΔT.

The material properties are considered functionally graded through the beam thickness. In the literature, various types of functions are used to describe how the material properties of FGM changes. In this paper, an exponential function is used to describe variation of the material properties through the thickness of the beam. Apart from Poisson's ratio (ν) which is considered to be constant, other material properties of the beam such as Young modulus (E) and the thermal expansion coefficient (α) are considered as [23]

$$E(z)=A_E e^{B_E(z+h)} \tag{2a}$$

$$\alpha(z)=A_\alpha e^{B_\alpha(z+h)} \tag{2b}$$

in which, A_E, A_α, B_E and B_α are defined as

$$A_E=E_c \tag{3a}$$

$$A_\alpha=\alpha_c \tag{3b}$$

$$B_E=\frac{1}{2h}\ln\left(\frac{E_m}{E_c}\right) \tag{3c}$$

$$B_\alpha=\frac{1}{2h}\ln\left(\frac{\alpha_m}{\alpha_c}\right) \tag{3d}$$

where subscripts m and c refer to metallic and ceramic constituents of FG material, respectively. According to the classical beam theory (CBT) the nonlinear strain field which presents large deflection in the Cartesian coordinate system, is written as

$$\varepsilon=\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\varepsilon_{th} \tag{4}$$

in which u and w are axial and transverse displacement of the beam, respectively and ε_{th} is thermal strain. Defining the force resultant, N_x, and moment resultant, M_x, per unit length as

$$(N_x,\ M_x)=b\int_{-h/2}^{h/2}(1,\ z)\sigma_{xx}(z)dz \tag{5}$$

and applying generalized Hooke's law, one may obtain N_x and M_x as

$$N_x=A_{11}\left(\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right)+B_{11}\frac{d^2w}{dx^2}+N_{th} \tag{6a}$$

$$M_x=B_{11}\left(\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right)+D_{11}\frac{d^2w}{dx^2}+M_{th} \tag{6b}$$

N_{th} and M_{th} are resultant thermal force and moment per unit length, respectively, which are defined as

$$(N_{th},\ M_{th})=b\iint(1,\ z)\frac{E(z)}{1-\upsilon^2}\alpha(z)\Delta T\ dA \tag{7}$$

And the other parameters A_{11}, B_{11} and D_{11} are defined as

$$(A_{11},B_{11},D_{11})=b\iint\frac{E(z)}{1-\upsilon^2}(1,z,z^2)dA \tag{8}$$

Integrating Eqs. (7) and (8) for the previously defined tapered FG beam, one can rewrite A_{11}, B_{11} and D_{11}, N_{th} and M_{th} as

$$A_{11}=C_{A_{11}}h(x) \tag{9a}$$

$$B_{11}=C_{B_{11}}h(x)^2 \tag{9b}$$

$$D_{11}=C_{D_{11}}h(x)^3 \tag{9c}$$

$$N_{th}=C_{N_{th}}h(x) \tag{9d}$$

$$M_{th}=C_{M_{th}}h(x)^2 \tag{9e}$$

where the constants $C_{A_{11}}$, $C_{B_{11}}$, $C_{D_{11}}$, $C_{N_{th}}$ and $C_{M_{th}}$ are defined in Appendix A.

Using the energy principle and CBT, one can derive the equilibrium equations as

$$\frac{dN_x}{dx}=0 \tag{10a}$$

$$\frac{d^2M_x}{dx^2}-N_x\frac{d^2w}{dx^2}=q \tag{10b}$$

From (10a), it can be concluded that N_x is constant and therefore can be considered as

$$N_x=K \tag{11}$$

where K is the unknown constant axial force in the beam to be determined. Substituting Eq. (11) into (6a) leads to

$$\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2=\frac{1}{A_{11}}\left(K-B_{11}\frac{d^2w}{dx^2}-N_{th}\right) \tag{12}$$

Based on (12), bending moment of the beam can be determined as

$$M_x=\frac{B_{11}}{A_{11}}\left(K-B_{11}\frac{d^2w}{dx^2}-N_{th}\right)+D_{11}\frac{d^2w}{dx^2}+M_{th} \tag{13}$$

Substitution of the constants defined in (9a)–(9e) into (13) leads to

$$M_x=h^3\frac{d^2w}{dx^2}\left(C_{D_{11}}-\frac{C_{B_{11}}^2}{C_{A_{11}}}\right)+h^2\left(C_{M_{th}}-\frac{C_{B_{11}}C_{N_{th}}}{C_{A_{11}}}\right)+h\frac{KC_{B_{11}}}{C_{A_{11}}} \tag{14}$$