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Self-similar solutions and converging shocks in a non-ideal gas with dust particles

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ABSTRACT

In this paper, we have used the Lie group of transformations and obtained the whole range of self-similar solutions to the problem of propagation of shock waves through a non-ideal, dusty gas. The conditions essential for the existence of similarity solutions for a strong shock are discussed. The problem of imploding (converging) shock wave is also worked out and the effects of the mass concentration of the dust particles, ratio of the density of solid particles to that of initial density of the medium, the relative specific heat and the effect of the non-ideal parameter, on the shock formation has been studied in detail.

1. Introduction

Lie group of transformations has been widely used to study the continuous symmetry in mathematics, theoretical physics and mechanics. It helps in simplifying the complicated problems of the physical system into solvable mathematical equations. The expanded Lie Group of transformations of partial differential equations means a continuous group of transformations, which act on the expanded space of variables and includes the equation parameters along with independent and dependent variables. Latest works on Lie groups and its applications in various fields can be found in [1–4]. Radha and Sharma [5] have used the Lie group of transformation method described in the said works and obtained the entire class of self-similar solutions for converging shocks in a relaxing gas. This method helps us to identify the medium for which the problem is invariant and admits self-similar solutions. Ames and Donato studied the evolution of weak discontinuities in a state characterized by invariant solutions [6] and Donato studied the similarity solutions and strong shocks in extended thermodynamics of rarefied gas [7]. Oliveri and Speciale used Lie group analysis to find exact solutions to the unsteady equations of perfect gases in [8] and exact solutions to the ideal magnetogasdynamics equations in [9]. The Lie group of transformation method was used to study the shock wave propagation through a dusty gas mixture obeying the equation of state of Mie-Gruneisen type in [10], solution of system of equations describing viscoelastic materials [11], interaction of a weak discontinuity wave with a bore in shallow water [12], interaction of discontinuous waves in a relaxing gas [13], self-similar shocks in a gas with dust particles [14] and self-similar solutions in a plasma with axial magnetic field (θ -pinch) [15].

For the system of quasilinear hyperbolic partial differential equations, it is hard, in general to determine a solution without approximations. Here, we assume that there exists a solution of the basic equations subject to the jump conditions along a set of curves called the similarity curves, for which the system of partial differential equations transforms to a set of ordinary differential equations. In the self-similar motion the flow variables describing the motion, i.e. velocity $u(x,t)$, density $\rho(x,t)$, pressure $p(x,t)$, etc. do not depend upon the x coordinate and time coordinate t separately but are functions only of their combination. (see [16]). Self-similar solutions are of two types. For the solutions of the first type the similarity exponent δ is determined either by dimensional considerations or from the conservation laws, and in the self-similar problems of the second type, the exponent δ cannot be found from the same without solving the equations. In such a situation the similarity exponent is determined by integrating the ordinary differential equations for the reduced functions (see [17]). Our present work is concerned with self-similar solutions of second type.

We consider a system of partial differential equations describing the one dimensional unsteady plane and radially symmetric flow of an inviscid gas with dust particles. It is assumed that the gas consists of a non-ideal gas and small solid particles. The present work is concerned with the cases when the mass concentration of the particles is generally same as that of the gas. The volume occupied by the particles is negligible because the

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density of the solid particles is much larger than that of the gas. The constants occurring in the expressions for the generators of the local Lie group of transformations are responsible for the different cases of possible solutions like power law, exponential or logarithmic shock paths. A particular case of the collapse of an imploding shock is studied for radially symmetric flows. For the flow variables which are unique to the medium, the self-similar exponent is determined using the assumption that solution of the system of ordinary differential equations which represent the self-similar motion is regular on a regular characteristic passing through the center (axis) of implosion. Numerical calculations have been carried out to determine the values of the self-similarity exponent and the profiles of the flow variables behind the shock; the values of the similarity exponent so obtained agree with those obtained using characteristic rule [10]. The influences of dust particles and Van der Waals excluded volume on the shock wave and the flow parameters behind the shock are studied.

2. Basic equations

The basic equations are based on account of the following assumptions (see [18–21]) (i) the gas is non-ideal and the specific heats are constant, (ii) the particles are spherical, of uniform size, incompressible and occupy less than 5% of the total volume, their specific heat is constant and the temperature is uniform within each particle, interaction between particles of different sizes is not considered, (iii) the flow is taken to be one dimensional, (iv) the particles are uniformly distributed over the cross section of the duct, and size and average spacing of the particles are small compared with the cross-sectional dimensions of the duct, (v) heat transfer and boundary layer effects with the duct walls are not considered, (vi) the effect of the particles on the gas appears at first in the wake of the particles and is then distributed over the rest of the gas by mixing, (vii) the pressure id not affected by the particles, (viii) external forces are not applied on the mixture, (ix) no mass transfer takes place between the two phases. The system equations that represent a planar ($m=0$), cylindrically symmetric ($m=1$) or a spherically symmetric ($m=2$) motion of a non-ideal gas with dust particles is given by [24]

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}+\frac{m\rho u}{x}=0,\\
&\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x}=0,\\
&\frac{\partial E_m}{\partial t}+u\frac{\partial E_m}{\partial x}-\frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)=0,
\end{aligned}\tag{1}$$

where u is the particle velocity along x-axis, t the time, ρ the density, p the pressure and E_m is the internal energy per unit mass of the mixture which is given by

$$E_m=\frac{(1-Z)(1-\tilde{b}\rho)p}{(\Gamma-1)\rho}.\tag{2}$$

Here, $Z=V_{sp}/V_g$ is the volume fraction and $k_p=m_{sp}/m_g$ is the mass fraction of the solid particles in the mixture where m_{sp} and V_{sp} are the total mass and volumetric extension of the solid particles and V_g and m_g are the total volume and total mass of the mixture; $\tilde{b}=b(1-k_p)$ where b is the Van der Waals excluded volume and lies in the range $0.9\times10^{-3}\le b\le 1.1\times10^{-3}$ [23]; the Grüneisen coefficient $\Gamma=\gamma(1+\lambda\beta)/(1+\lambda\beta\gamma)$, $\lambda=k_p/(1-k_p)$, $\beta=c_{sp}/c_p$, $\gamma=c_p/c_v$ where c_{sp} is the specific heat of solid particles, c_p the specific heat of the gas at constant pressure, and c_v the specific heat of the gas at constant volume. The entities Z and k_p are related via the expression $Z=\theta\rho$, where $\theta=k_p/\rho_{sp}$, with ρ_{sp} as the species density of the solid particles. We introduce the variable $G=\rho_{sp}/\rho_g$, i.e. the ratio of the density of the solid particles to the species density of the gas, with respect to which the variations of the flow variables will be calculated and compared. The equation of state are given by

$$p=\frac{(1-k_p)}{(1-Z)(1-\tilde{b}\rho)}\rho RT,\tag{3}$$

where T is the temperature of the gas and that of the solid particles and R is the specific gas constant.

Using Eq. (3) and neglecting $O(b^2)$ terms, Eq. $(1)_3$ can be written as

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\rho C^2\left(\frac{\partial u}{\partial x}+\frac{mu}{x}\right)=0,\tag{4}$$

where

$$C=\left(\frac{(\Gamma-\alpha_2\rho^2)p}{(1-\alpha_1\rho+\alpha_2\rho^2)\rho}\right)^{1/2},\tag{5}$$

is the equilibrium speed of sound in the mixture with $\alpha_1=\theta+\tilde{b}$, $\alpha_2=\theta\tilde{b}$.

Depending upon the presence of the parameters θ and (or) b, the following cases can arise.

Case 1: If $\theta=0$ and $b=0$ then $\Gamma=\gamma$, $C^2=\gamma p/\rho$ and the mixture becomes an ideal gas (ideal in the sense that the particle interactions are absent).
Case 2: If $\theta=0$, $b\neq0$, then $\Gamma=\gamma$, $C^2=\gamma p/\rho(1-\tilde{b}\rho)$ and the mixture is a non-ideal (the Van der Waal) gas.
Case 3: If $\theta\neq0$, $b=0$, then $C^2=\Gamma p/\rho(1-\theta\rho)$ and the case is a mixture of an ideal gas with dust particles.
Case 4: If $\theta\neq0$, $b\neq0$, then C is given by (5) and the case is a mixture of a non-ideal (Van der Waals) gas with dust particles.

If the system of equations are written in the conservation form

$$G_t(x,t,U)+F_x(x,t,U)=H(x,t,U),\tag{6}$$

where G and F and H are column vectors having n components, then the Rankine–Hugoniot equations for shock waves are given by

$$|[G_i]|V=|[F_i]|,\quad i=1,2,\ldots n\tag{7}$$

where V is the shock velocity. Here, $|[X]|=X-X_0$ is the jump in the variable X, where the variables in the medium ahead of the shock are referred by the subscript 0 and the medium behind the shock are without any subscript.

In the conservation form (6) Eqs. $(1)_{1,2}$ and (4) can be written with following forms of G, F and H

$$\begin{aligned}
G&=\left(\rho,\rho u,\frac{(1-\alpha_1\rho+\alpha_2\rho^2)}{(\Gamma-1)}p+\frac{\rho u^2}{2}\right)^{tr},\\
F&=\left(\rho u,\rho u^2+p,u\left(\frac{(1-\alpha_1\rho+\alpha_2\rho^2)}{(\Gamma-1)}p+p+\frac{\rho u^2}{2}\right)\right)^{tr},\\
H&=\left(-\frac{m\rho u}{x},-\frac{m\rho u^2}{x},-\frac{m\rho u^3}{2x}-\frac{mpu}{x}\left(\frac{(\Gamma-\alpha_1\rho+\alpha_2\rho^2)}{(\Gamma-1)}\right)\right)^{tr}.
\end{aligned}\tag{8}$$

It is assumed that the shock front $x=\chi(t)$, is moving with the velocity $V=\dot{\chi}(t)$ into the inhomogeneous medium given by $u_0=0$, $p_0=$ constant and $\rho_0=\rho_0(x)$. From the Rankine–Hugoniot jump conditions (7) and the conservation form (6), it has been observed that the density ρ across the shock front satisfy the following equation:

$$\begin{aligned}
&\rho_0V^2\left(1-\frac{\rho_0}{\rho}\right)\{(\Gamma-1)\rho-(\Gamma+1)\rho_0+2\alpha_1\rho\rho_0-2\alpha_2\rho_0\rho^2\}\\
&\quad+2p_0\{\rho(\Gamma-\alpha_1\rho_0+\alpha_2\rho_0^2)-\rho_0(\Gamma-\alpha_1\rho+\alpha_2\rho^2)\}=0.
\end{aligned}\tag{9}$$

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