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Bouncing ball dynamics: Simple model of motion of the table and sinusoidal motion

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ABSTRACT

Non-linear dynamics of a bouncing ball moving vertically in a gravitational field and colliding with a moving limiter is considered. The Poincaré map, describing evolution from an impact to the next impact, is used to analyse the original system. Sinusoidal displacement of the table, defining the standard model, is approximated in one period of the limiter's motion by a cubic spline, thus making analytical computations possible. Analytical and numerical results, based on Implicit Function Theorem, obtained for this simplified model, are used to elucidate dynamics of the standard model of the bouncing ball. Finally, the same techniques are applied to investigate dynamics of the standard model.

1. Introduction

In the present paper we study dynamics of a ball moving vertically in a gravitational field and impacting with a periodically moving limiter (a table). This model belongs to the field of non-smooth and non-linear dynamical systems [1–4]. In such systems non-standard bifurcations such as border-collisions and grazing impacts leading often to complex chaotic motions are typically present. It is important that non-smooth systems have many applications in technology [5–10].

Impacting systems studied in the literature can be divided into three main classes: bouncing ball models [11–13], impacting oscillators [14] and impacting pendulums [15,10], see also [1]. In dynamics with impacts it is usually difficult or even impossible to solve non-linear equation for an instant of the next impact. For example, in the bouncing ball models the table's motion has been usually assumed to be in a sinusoidal form, cf. [13] and references therein. This choice of the limiter's motion leads indeed to a non-tractable non-linear equation for time of the next impact. To tackle this problem we proposed a sequence of models in which periodic motion of the table is assumed (in one period of limiter's motion) as a low-order polynomial of time [16]. It is thus possible to approximate the sinusoidal motion of the table more and more exactly and conduct analytical computations. Carrying out this plan we have studied several such models with linear, quadratic and cubic polynomials [17–20].

In the present work we conduct analytical and numerical investigations of the model in which sinusoidal displacement of the table is approximated in one period by four cubic polynomials. We shall refer to this model as $\mathcal{M}_C$. The reason to use this approximation is that it is much more exact than that used in Ref. [20] and still allows analytical computations.

Simultaneously, we study the standard dynamics of bouncing ball with sinusoidal motion of the limiter, referred to as $\mathcal{M}_S$. We are using techniques based on the Implicit Function Theorem [21] which can be applied to both models. It should be stressed that results obtained for the model $\mathcal{M}_S$ can be compared with experimental studies, see [22–24] for the early papers, summarized in [25], and [26] for recent work.

The paper is organized as follows. In Section 2 a one dimensional dynamics of a ball moving in a gravitational field and colliding with a table is reviewed and the corresponding Poincaré map is described. Two models of the limiter's motion $\mathcal{M}_C$ and $\mathcal{M}_S$ are next defined. Bifurcation diagrams are computed for $\mathcal{M}_C$ and $\mathcal{M}_S$. In Sections 3–6 a combination of analytical and numerical approach is used to investigate selected problems of dynamics in models $\mathcal{M}_C$ and $\mathcal{M}_S$. More exactly, birth of low velocity n-cycles is investigated in Section 3 and birth of high velocity 3-cycles is studied in Section 4 for both models. In Section 5 the case of N impacts in one interval of the limiter's motion is studied for the model $\mathcal{M}_C$ while in Section 6 we study launching mechanism and mixing for the model $\mathcal{M}_S$. We summarize our results in the last section.

2. Bouncing ball: a simple motion of the table

Let a ball moves vertically in a constant gravitational field and collides with a periodically moving table. We treat the ball as a

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material point (this condition has been relaxed in [27]) and assume that the limiter's mass is so large that its motion is not affected at impacts. Dynamics of the ball from an impact to the next impact can be described by the following Poincaré map in a non-dimensional form [28] (see also Ref. [12] where analogous map was derived earlier and Ref. [13] for generalizations of the bouncing ball model):

$$\gamma Y(T_{i+1}) = \gamma Y(T_i) - \Delta_{i+1}^2 + \Delta_{i+1} V_i, \tag{1a}$$

$$V_{i+1} = -RV_i + 2R\Delta_{i+1} + \gamma(1+R)\dot{Y}(T_{i+1}), \tag{1b}$$

where T_i denotes the time of the i-th impact and V_i is the corresponding post-impact velocity while $\Delta_{i+1} \equiv T_{i+1} - T_i$. The parameters γ and R are a non-dimensional acceleration and the coefficient of restitution, $0 \le R < 1$ [5], respectively and the function $Y(T)$ represents the limiter's motion. The limiter's motion has been typically assumed to be in the sinusoidal form, $Y_S(T) = \sin(2\pi T)$. Eq. (1) and $Y = Y_S$ lead to the model $\mathcal{M}_S$. This choice of limiter's motion leads to serious difficulties in solving the first of (1) for T_{i+1}, thus making analytical investigations of dynamics hardly possible. Accordingly, we have decided to simplify the limiter's periodic motion to make (1a) solvable. The function

$$Y_C(T) = \begin{cases} f_1(T), & 0 \le \hat{T} < \frac{1}{4} \\ f_2(T), & \frac{1}{4} \le \hat{T} < \frac{1}{2} \\ f_3(T), & \frac{1}{2} \le \hat{T} < \frac{3}{4} \\ f_4(T), & \frac{3}{4} \le \hat{T} \le 1 \end{cases} \tag{2}$$

$$f_1(T) = (32\pi - 128)\hat{T}^3 + (-16\pi + 48)\hat{T}^2 + 2\pi\hat{T} \tag{3a}$$

$$f_2(T) = (128 - 32\pi)\hat{T}^3 + (-144 + 32\pi)\hat{T}^2 + (48 - 10\pi)\hat{T} - 4 + \pi \tag{3b}$$

$$f_3(T) = (128 - 32\pi)\hat{T}^3 + (-240 + 64\pi)\hat{T}^2 + (144 - 42\pi)\hat{T} - 28 + 9\pi \tag{3c}$$

$$f_4(T) = (32\pi - 128)\hat{T}^3 + (336 - 80\pi)\hat{T}^2 + (-288 + 66\pi)\hat{T} + 80 - 18\pi \tag{3d}$$

approximates $Y_S = \sin(2\pi T)$ on the intervals $[k, k+1]$, $k = 0, 1, \ldots$, with $\hat{T} = T - \lfloor T \rfloor$, where $\lfloor x \rfloor$ is the floor function – the largest integer less than or equal to x. The model $\mathcal{M}_C$ consists of Eqs. (1), (2), and (3) with control parameters R and γ. We shall also need velocities of the limiter, defined as $g_i(T) \overset{df}{=} (d/dt) f_i(T)$, $i = 1, \ldots, 4$.

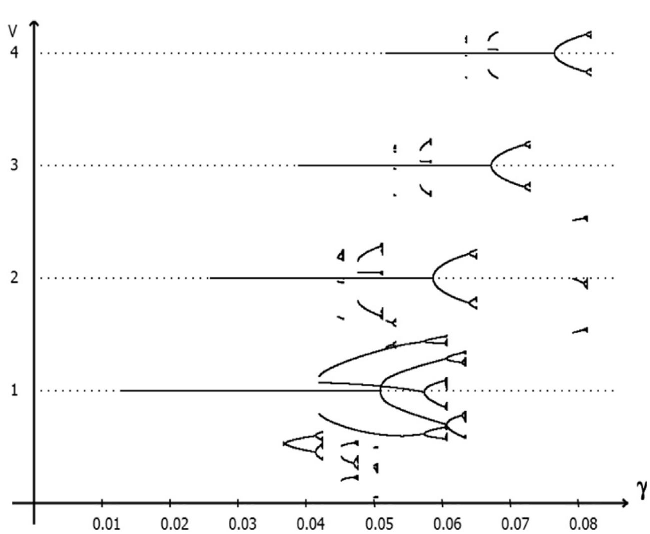

Fig. 1. Bifurcation diagram for the model $\mathcal{M}_C$, $R = 0.85$: velocities after impacts versus control parameter γ.

Comparison of Figs. 2 and 3 from Ref. [16] shows that this approximation is much better that one cubic polynomial approximation investigated in Ref. [20].

In Fig. 1 we show the bifurcation diagram with velocities after impacts versus γ computed for growing γ and $R = 0.85$. It follows that dynamical system $\mathcal{M}_C$ has several classes of attractors: fixed points period-doubling to chaos, small velocity k-cycles, high-velocity 3-cycles and some other attractors including grazing manifold (not shown in the figure). We shall investigate some of these attractors in the next sections combining analytical and numerical approaches (general analytical conditions for birth of new modes of motion were given in [29]).

In Fig. 2 the corresponding bifurcation diagram for the sinusoidal motion is shown. Similarity of Figs. 1 and 2 suggests that analytical results obtained for the model $\mathcal{M}_C$ will shed light on the problem of sinusoidal motion, $\mathcal{M}_S$.

We realize finally that the approximation defined in Eqs. (2) and (3) leads to significant improvement on one cubic polynomial approximation, cf. Fig. 1 from Ref. [20] where the corresponding bifurcation diagram is shown.

3. Birth of low velocity *k*-cycles

In this section we shall study birth of low velocity k-cycles which can be seen in the bifurcation diagrams, Figs. 1 and 2, for $\gamma > 0.03$ and $V < 1$. In the case of such cycles $T_1, T_2, \ldots, T_k \in (0, 1)$ and $T_{k+1} - 1 = T_1$. Of course, it is possible to follow periodic orbits backwards, i.e. iterating the map (1) until the convergence to the k-cycle is achieved for some initial condition and some γ. Then the value of γ is decreased (slightly) and the map is iterated again (until convergence is obtained) with the previously computed k-cycle as the initial condition. This method although leads to the determination of the critical value of γ at which the k-cycle disappears for decreasing γ (or is born for growing γ) but is time-consuming and not very effective due to very poor convergence near the threshold.

On the other hand, analytical conditions for birth of k-cycles are found below. In what follows theorems about differentiation of implicit functions [21] will turn out to be useful since Eq. (1a) defines T_{i+1} implicitly. Consider equation

$$F(T_1, T_2) = 0, \tag{4}$$

which defines dependence of, say, T_2 on T_1, see [21] where necessary and sufficient assumptions are given. Then it follows from implicit function theorem that

$$\frac{dT_2}{dT_1} = -\frac{F_1'}{F_2'}. \tag{5}$$

where $F_1' \equiv \partial F/\partial T_1$, $F_2' \equiv \partial F/\partial T_2$.

In a more complicated case, equations

$$F(T_1, T_2, T_3) = 0, \quad G(T_1, T_2, T_3) = 0, \tag{6}$$

define T_2 and T_3 as functions of T_1 under appropriate assumptions. We can now compute derivatives with respect to T_1 as [21]

$$\frac{\partial T_2}{\partial T_1} = -\frac{\det\begin{pmatrix} F_1' & G_1' \\ F_3' & G_3' \end{pmatrix}}{\det\begin{pmatrix} F_2' & G_2' \\ F_3' & G_3' \end{pmatrix}}, \quad \frac{\partial T_3}{\partial T_1} = -\frac{\det\begin{pmatrix} F_2' & G_2' \\ F_1' & G_1' \end{pmatrix}}{\det\begin{pmatrix} F_2' & G_2' \\ F_3' & G_3' \end{pmatrix}}, \tag{7}$$

with $F_1' \equiv \partial F/\partial T_1$, $F_2' \equiv \partial F/\partial T_2$, $F_3' \equiv \partial F/\partial T_3$ and analogous notation for G_i', $i = 1, 2, 3$.

3.1. *Low velocity 2-cycle in the model $\mathcal{M}_C$*

Numerical tests show that a 2-cycle fulfilling conditions $T_1 \in (0, \frac{1}{4})$, $T_2 \in (\frac{1}{2}, \frac{3}{4})$ and $T_3 = T_1 + 1$ is stable. This 2-cycle can