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Constructing transient response probability density of non-linear system through complex fractional moments

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ABSTRACT

The probability density function for transient response of non-linear stochastic system is investigated through the stochastic averaging and Mellin transform. The stochastic averaging based on the generalized harmonic functions is adopted to reduce the system dimension and derive the one-dimensional Itô stochastic differential equation with respect to amplitude response. To solve the Fokker–Plank–Kolmogorov equation governing the amplitude response probability density, the Mellin transform is first implemented to obtain the differential relation of complex fractional moments. Combining the expansion form of transient probability density with respect to complex fractional moments and the differential relations at different transform parameters yields a set of closed-form first-order ordinary differential equations. The complex fractional moments which are determined by the solution of the above equations can be used to directly construct the probability density function of system response. Numerical results for a van der Pol oscillator subject to stochastically external and parametric excitations are given to illustrate the application, the convergence and the precision of the proposed procedure.

1. Introduction

Random response prediction of non-linear system is one of the fundamental and important topics in stochastic dynamics. All the statistical information of the system response can be derived by the response probability density function, and so the establish of response probability density is the ultimate target of random response analysis. The analysis on the stationary probability density functions of system responses is relatively easy and the exact and approximate solutions have been widely studied [1–5]. The obtained stationary functions, however, can only give the statistical information of stationary responses. The prediction of transient probability density of random responses is a challenging project due to its dependence on time.

Only for very special first-order non-linear systems, the exact probability density of transient responses can be obtained [6–8]. Thus, some approximate methods, such as Monte Carlo simulation, path integral approach, perturbation technique and Galerkin method, have been developed for the general first-order and second-order non-linear stochastic systems [2,9–11]. Monte Carlo simulation directly handles the original equation and the benefit is the versatility to system and excitation property, while the drawbacks are the long computing time and high cost. Path integral approach, perturbation technique and Galerkin method are adopted to approximately solve the Fokker–Plank–Kolmogorov (FPK) equation governing the transient response probability density. Path integral approach is also a numerical method. Although both of perturbation technique and Galerkin method provide the semi-analytical solution of the FPK equation, the former is restricted to cases with weak non-linearity while the latter, in a certain extent, breaks this restriction [12,13]. When using the Galerkin method, the solution of FPK equation is approximately expressed by a series in terms of a set of properly state-dependent orthogonal basis functions with time-dependent coefficients which are calculated by making the projection of the residual error vanish on a proper set of independent functions. The precision of the Galerkin method heavily relies on the selection of the basis functions. In addition, the application of Galerkin method to solve the FPK equation is mainly limited by the system dimension. For high-dimensional system, it is quite difficult to derive the semi-analytical result through this method. The combined application of the stochastic averaging technique and the FPK equation method is a powerful tool to investigate the system response to random excitations, and the stochastic averaging is used to reduce system dimension and derive lower-dimensional FPK equation. Thus, the combination of stochastic averaging and

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Galerkin method can deal with systems of higher dimensions [14–16].

Noted that in the fractional calculus in complex domain, Mellin transform of the probability density function is linearly related to complex fractional moment and a new expansion form of the probability density function with respect to complex fractional moments has been obtained [17,18]. Mellin transform and the new form of expansion can be used to solve the FPK equation and to determine the probability density function for transient response of a first-order non-linear system excited by a Gaussian white noise [19]. Compared to the Galerkin method, this procedure is based on the complex fractional moments which indicates the character of the system and do not need to select the basis functions. Theoretically, this procedure can be directly generalized to solve two- dimensional FPK equation, the complexity, however, will be exponentially increased. This severely limits its application on mechanical and structural systems. Similar to the combination of Galerkin method and stochastic averaging, the combined applications of Mellin transform and stochastic averaging may be an effective technique to derive the transient response probability density.

In the present manuscript, the transient response probability density of the second-order non-linear system subject to stochastic external and parametric excitations is investigated. By using the stochastic averaging based on the generalized harmonic functions, the FPK equation governing the amplitude response probability density is derived. Taking Mellin transform on the FPK equation and referring to the expansion form of the amplitude response probability density yield a set of linear ordinary differential equations governing the complex fractional moments. Finally, the transient probability density is constructed through the complex fractional moments. Numerical results for a representative example are given to illustrate the application, the convergence and the precision of the proposed procedure. The influence of the non-quiescent initial condition on the evolution of transient probability density is briefly discussed.

2. Stochastic averaging for system reduction

The one-degree-of-freedom oscillator with non-linear damping subject to externally and parametrically random excitations is frequently encountered in mechanical and structural engineering, and the researches on transient response aspect are of great significance. The equation of motion of the non-linear stochastic oscillator is as follows,

$$\ddot{X}+f(X,\dot{X})\dot{X}+\omega^2X=g_i(X,\dot{X})W_i(t) \tag{1}$$

in which X is the system displacement, $\dot{X}$ is the velocity, $f(X,\dot{X})$ is the non-linear damping coefficient, ω is the natural frequency, $W_i(t)$ are independent Gaussian white noises with intensities $2D_i$ and $g_i(X,\dot{X})$ are the magnitudes of external and parametric excitations. The non-linear damping coefficient and the excitation magnitudes are supposed as ε order and $\varepsilon^{1/2}$ order, respectively, *i.e.*, the considered system is with light damping and weak excitations. Note that an arbitrary function can be described by polynomial through Taylor expansion, so without loss of generality, $f(X,\dot{X})$ and $g_i(X,\dot{X})$ are confined to be the polynomials of displacement X and velocity $\dot{X}$.

According to the quasi-conservative property, it is reasonable to suppose that system (1) has a family of quasi-periodic solutions surrounding the origin of the phase plane. Introduce the following van der Pol transformation [20],

$$X=A\cos\Theta(t)$$
$$\dot{X}=-A\omega\sin\Theta(t) \tag{2}$$

where $\Theta(t)=\omega t+\Gamma(t)$. Due to the small parameter assumptions on damping and excitation amplitudes, the system amplitude A and initial phase Γ are slowly varying processes.

Accomplishing the van der Pol transformation in Eq. (2) yields the stochastic differential equations governing the system amplitude A and initial phase Γ. Furthermore, the slowly varying process $A(t)$ converges weakly into a diffusion Markov process, and the limiting process is described by the following averaged Itô stochastic differential equation through the Stratonovich-Khasminskii limit theorem [21–23],

$$dA=m(A)dt+\sigma(A)dB(t) \tag{3}$$

where the drift and diffusive coefficients are,

$$m(A)=\langle -Af\sin^2\Theta\rangle_\Theta+D_i\left\langle \frac{\partial g_i}{\partial A}\frac{\sin\Theta g_i\sin\Theta}{\omega^2}+\frac{\partial g_i}{\partial \Gamma}\frac{\sin\Theta g_i\cos\Theta}{A\omega^2}\right\rangle_\Theta$$
$$\sigma^2(A)=2D_i\left\langle\frac{g_i^2\sin^2\Theta}{\omega^2}\right\rangle_\Theta \tag{4}$$

The FPK equation associated with the averaged Itô stochastic differential Eq. (3), governing the evolution of the amplitude response probability density, is of the following form,

$$\frac{\partial p(A,t)}{\partial t}=-\frac{\partial}{\partial A}[m(A)p(A,t)]+\frac{1}{2}\frac{\partial^2}{\partial A^2}[\sigma^2(A)p(A,t)] \tag{5}$$

As $f(X,\dot{X})$ and $g_i(X,\dot{X})$ are of polynomial form, the drift and diffusive coefficients can be simplified to the following form:

$$m(A)=a_{-1}A^{-1}+\sum_{i=1}^{k_1}a_iA^i$$
$$\sigma^2(A)=b_0+\sum_{i=1}^{k_2}b_iA^i \tag{6}$$

where the coefficients a_i and b_i are related to the system parameters and excitation parameters.

3. Transient probability density of amplitude response

Obviously, the direct solving of FPK Eq. (5) under given boundary and initial conditions will yield the transient probability density of amplitude response. As mentioned above, the complex fractional moment method is an effective technique to solve the one-dimensional FPK equation. The FPK Eq. (5) with respect to amplitude response probability density is just one-dimensional, so can be considered to be solved by complex fractional moments. To the sake of the descriptive integrality, some fundamental concepts are briefly illustrated.

3.1. Fundamental concepts on Mellin transform and complex fractional moment

Suppose $q(x)$ is a real function defined in $0\le x<\infty$. The Mellin transform is defined as follows [19]:

$$M_q(\gamma-1)=\int_0^\infty q(x)x^{\gamma-1}dx \tag{7}$$

where $\gamma=\rho+I\eta$ is a complex number and I is the imaginary unit. If the Mellin transform exists, then $q(x)$ may be restituted by the following formula:

$$q(x)=\frac{1}{2\pi}\int_{\eta=-\infty}^{\infty}M_q(\gamma-1)x^{-\gamma}d\eta,\quad x>0 \tag{8}$$

The integration in Eq. (8) is performed along the imaginary axis η under the fixed real part ρ.

The condition for the existence of the Mellin transform is $-\rho_l<\rho<-\rho_h$, in which ρ_l and ρ_h satisfy $q(x)=O(x^{\rho_l})$ as $x\to 0$ and