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Deformation and stability of a pinned shallow arch constrained by a rigid plate and loaded by a concentrated moment

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A B S T R A C T

This paper studies the behavior of a pinned half-sine arch, with a center rigid constraint plate, under a static concentrated moment. Under proper loading conditions, the arch will be in contact with the constraint plate at discrete points. This type of configurations is referred to as the contact equilibrium configuration. Geometric restrictions on the deformation of the arch at the contact point are derived. Then, the method of mode expansion is used to solve the force equilibrium equations together with the geometric restrictions for the equilibrium configuration. Due to the restrictions on the deformation of the arch imposed by the constraint plate, the classical potential energy method cannot be directly applied to determine the stability of the contact equilibrium configuration. A modified potential energy method is proposed for overcoming this problem. With the proposed method, the effects of the magnitude and location of the applied moment on the deformation and stability of the arch are investigated thoroughly. We find that, in the presence of the constraint plate, the arch possesses more complicated deformation patterns. Finally, experiments are conducted to validate the theoretical results.

1. Introduction

It is well known that an unconstrained arch may jump from one stable equilibrium configuration to another at a critical level of load. This process is called snap-through buckling. The quick (dramatic) change in configuration in a snap-through buckling has found applications in different fields of sensors and actuators [1–7]. However, in some applications, a constraint, in the form of a rigid plate along the line connecting the two ends of the arch, is presented so that some of the stable configurations are not accessible. A mechanical button on an instrument panel can be qualitatively regarded as a shallow arch with a rigid constraint plate underneath. For an electrically actuated micro-arch, the electrode beneath the arch can be treated as a rigid constraint plate [8,9]. A vehicle traveling on a concrete pavement blowup presents another practical example [10]. In this case, the shallow arch represents the blowup and the rigid plate the foundation. The presence of the constraint plate may result in complicated deformation patterns and poses some challenges for the stability analysis, especially when the arch touches the constraint plate.

Previous works on snap-through buckling can be divided into two groups: static snap-through buckling and dynamic snap-through buckling. For static snap-through buckling, the lateral load is applied quasi-statically. In this case, emphasis is put on the determination of the critical load and the dependence of the critical load on various system parameters [11–20]. For dynamic snap-through buckling, the lateral load is time varying and the focus is put on the conditions under which dynamic snap-through buckling will not happen [21–34]. Snap-through buckling for an unconstrained arch has been investigated thoroughly. By contrast, relatively little research has been devoted to the behavior of a shallow arch with a center constraint plate. This paper aims to investigate the effects of the center constraint plate on the post-buckling behavior of a shallow arch.

The load applied to a mechanical button or the weight of a vehicle on a pavement blowup can be treated as a concentrated force. On the other hand, the effect of a piezoelectric patch, which is often used as an actuator to drive arch-like mini-structures, can be modeled as a pair of concentrated moments [35]. The behavior of an elastica constrained by a flat surface and subjected a point load has been investigated by Chen and Wu [36]. In this paper, we study the deformation and stability of a shallow arch with a center rigid constraint plate and loaded by a concentrated moment.

The rest of this paper is organized as follows. Section 2 presents the mathematical model and analysis procedure. The modified potential energy method for the determination of the stability of the contact equilibrium configuration is described in detail. Section 3 introduces the experimental setup. In Section 4, the

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variation of the deformation patterns and the associated stability with the applied load are studied both theoretically and experimentally. Finally, Section 5 offers brief conclusions.

2. Analysis

2.1. Equilibrium equation

Consider the shallow arch shown in Fig. 1. It is homogeneous and has uniform cross-section. The ends are pinned at a distance l. A smooth rigid plate is placed between the ends along the $\bar{x}$-axis. On the basis of the Euler beam theory, the equilibrium equation of the loaded arch can be written as

$$EI\left(\frac{\partial^4\bar{y}}{\partial\bar{x}^4}-\frac{\partial^4\bar{y}_0}{\partial\bar{x}^4}\right)+H\frac{\partial^2\bar{y}}{\partial\bar{x}^2}=-\bar{q} \tag{1}$$

where E is Young's modulus, I the moment of inertia of the cross-section, $\bar{y}$ and $\bar{y}_0$ respectively the coordinates of the deformed and the initial centerlines measured from the $\bar{x}$-axis, H the axial force, $\bar{q}$ the distributed load. The axial force can be expressed as

$$H=\frac{EA}{2l}\int_0^l\left[\left(\frac{\partial\bar{y}_0}{\partial\bar{x}}\right)^2-\left(\frac{\partial\bar{y}}{\partial\bar{x}}\right)^2\right]d\bar{x}, \tag{2}$$

where A is the cross-sectional area.

For the sake of convenience of discussion, we introduce the following dimensionless parameters:

$$r=\sqrt{\frac{I}{A}},\quad y=\frac{\bar{y}}{2r},\quad x=\frac{\bar{x}\pi}{l},\quad P=\frac{H}{EI\pi^2/l^2},\quad q=\frac{l^4\bar{q}}{2\pi^4EIr}, \tag{3}$$

then Eqs. (1) and (2) respectively take the following forms

$$(y-y_0)''''+Py''=-q,\quad 0<x<\pi \tag{4}$$

with

$$P=\frac{2}{\pi}\int_0^\pi[(y_0')^2-(y')^2]dx, \tag{5}$$

where a prime indicates differentiation with respect to x.

The initial unloaded shape y_0 of the arch is assumed to be

$$y_0=h\sin(x) \tag{6}$$

with h representing the dimensionless central arch rise. With the pinned end conditions, the Fourier representation of the deformed shape y can be given in terms of sine functions as

$$\hat{y}(x)=\sum_{n=1}^{\infty}\hat{y}_n\sin(nx) \tag{7}$$

The load distribution q will also be expressed in a Fourier sine series

$$q(x)=\sum_{n=1}^{\infty}q_n\sin(nx) \tag{8}$$

In particular, for a concentrated moment applied at $x=d$,

$$q(x)=-M\delta'(x-d),$$

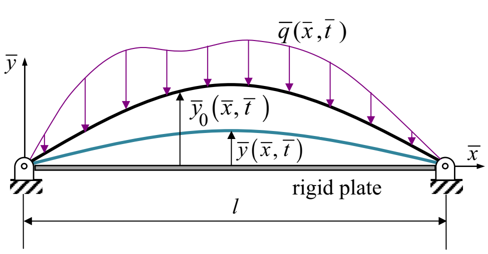

Fig. 1. Schematic diagram of a shallow arch constrained by a rigid plate.

and the associated Fourier coefficients are

$$q_n=2\pi M_0 n\cos nd$$

with $M_0=M/\pi^2$.

Substituting Eqs. (6)–(8) into Eqs. (4) and (5) and making use of the orthogonal property of the sine functions, we obtain the following set of equations of equilibrium

$$\begin{cases}(1-\hat{P})\hat{y}_1=-q_1+h\\ n^2(n^2-\hat{P})\hat{y}_n=-q_n,\quad n\geq 2\end{cases} \tag{9}$$

and

$$\hat{P}=h^2-\sum_{n=1}^{\infty}n^2\hat{y}_n^2 \tag{10}$$

2.2. Contact equilibrium configurations

Consider a concentrated moment of magnitude M applied at $x=d$. When the moment is increased gradually from zero, the deflection of the arch gradually increases. Initially, the arch is not in contact with the rigid plate. This type of equilibrium configuration is called the non-contact (equilibrium) configuration. As M reaches a critical value, either the lowest point of the deformed arch touches the rigid plate placed along the x-axis, or the non-contact configuration becomes unstable. In both cases, the arch will eventually be in contact with the rigid plate. For the convenience of discussion, the equilibrium configuration in contact with the rigid plate is henceforth referred to as the contact (equilibrium) configuration. It can be shown that, under the application of a concentrated moment, line contact between the arch and the rigid plate is not possible (a brief proof is given in the Appendix). Consequently, the arch can only be in contact with the rigid plate at discrete points. Assume that there is only one contact point at $x=s$ and the contact force is R (Fig. 2). In this case, the load distribution is

$$q(x)=-M\delta'(x-d)-R\delta(x-s),$$

and the associated Fourier coefficients are

$$q_n=2\pi M_0 n\cos nd-2R_0\sin ns, \tag{11}$$

where

$$M_0=\frac{M}{\pi^2}\quad\text{and}\quad R_0=\frac{R}{\pi}.$$

The geometric restrictions at the contact point are

$$y(s)=0\quad\text{and}\quad y'(s)=0 \tag{12}$$

or equivalently

$$\textstyle\sum_{n=1}\hat{y}_n\sin(ns)=0\quad\text{and}\quad\sum_{n=1}n\hat{y}_n\cos(ns)=0 \tag{13}$$

Eqs. (9), (10) and (13) form a complete set of equations for the unknowns, which includes the Fourier coefficients $\hat{y}_i$ of the deformed shape of the arch, axial force $\hat{P}$, location of the contact point s, and magnitude of the contact force R.

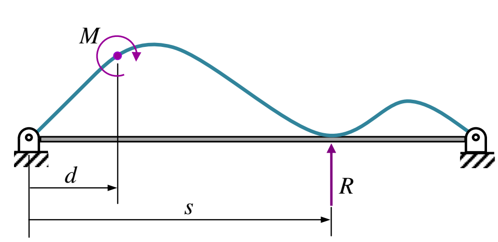

Fig. 2. Schematic diagram of a shallow arch with one contact point.