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Physically and geometrically non-linear vibrations of thin rectangular plates

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ABSTRACT

Static deflection as well as free and forced large-amplitude vibrations of thin rectangular rubber plates under uniformly distributed pressure are investigated. Both physical, through a neo-Hookean constitutive law to describe the non-linear elastic deformation of the material, and geometrical non-linearities are accounted for. The deflections of a thin initially flat plate are described by the von Kármán non-linear plate theory. A method for building a local model, which approximates the plate behavior around a deformed configuration, is proposed. This local model takes the form of a system of ordinary differential equations with quadratic and cubic non-linearities. The corresponding results are compared to the exact solution and are found to be accurate. Two models reflecting both physical and geometrical non-linearities and geometrical non-linearities only are compared. It is found that the sensitivity of the deflection to the physically induced non-linearities at moderate strains is significant.

1. Introduction

Shell-type structures are widely used in engineering and are frequently subjected to periodic external loadings. In the vast majority of applications, classical materials like steels are considered. Nevertheless, in modern bio-engineering for instance, numerical predictions require the simulation of biological materials involving more sophisticated properties. Not only the *geometrical* non-linearity (the non-linearity of strain–displacements relations), but also the *physical* or material non-linearity (non-linearity of stress-strain relations) should be addressed. Reviews on geometrically non-linear vibrations of shells and plates can be found in [1–4].

Several models describing the physical non-linearity aspect are available. A simple phenomenological model was developed by Kauderer, yet ignoring the framework of the widely accepted strain energy formulation [5]. This model is suitable for the description of non-linear elasticity of some metals, like copper for instance, and recent associated applications for dynamical problems of shell-type structures can be found in [6,7]. Nevertheless, models of *hyperelasticity* are preferred [5] for the description of rubber-like materials and soft biological tissues. However, due to the complicated nature of the corresponding derivations, only a few studies on the dynamics of shell and plate structures made of hyperelastic materials are reported [8,9]. A majority of these studies deal with simple geometries (spherical and circular cylindrical shells, circular membranes) and assume that the shape after deformation is identical. This allows for the derivation of closed-form expressions.

The first investigation on the radial oscillations of circular cylindrical tubes was undertaken by Knowles [10,11]. He considered tubes made of isotropic incompressible hyperelastic material and reduced the equations of motion to second order ordinary differential equations. In the same line, the paper by Akkas [12] deals with large-amplitude normal oscillations of a spherical membrane subjected to a uniform inflating step-pressure. It is found that a dynamic snap-out instability problem can occur for some values of the material coefficients. The relationship between the static behavior of the membrane and the dynamic instability is discussed; below the dynamic snap-out pressure threshold, the motion of the spherical membrane is always periodic. Finite-amplitude vibrations of non-linearly elastic incompressible spherical and circular cylindrical shells are mathematically examined by Calderer [13]. Due to the considered geometry it is assumed that the deformed structures are still a spherical or circular cylindrical shell. Under this assumption the radial periodic vibrations are investigated. Verron et al. [14] explored the dynamic inflation of spherical membranes of Mooney–Rivlin material for which a maximum in the frequency of oscillation versus pressure is detected.

Akyuz and Ertepinar [15] investigated cylindrical shells subjected to traction. Unlike other existing works in the literature, hyperelasticity

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is implemented through a Levinson–Burgess polynomial compressible material: this model is very similar to neo-Hookean one and coincides with it for incompressible materials. The static equilibrium in the finite deformation framework is first investigated, in the vicinity of which small-amplitude free vibrations are explored through small perturbation theory. The governing equations are solved numerically using a shooting method and the associated results compare well with the finite element solution. The condition of existence of periodic vibrations with respect to the geometrical and material shell properties is obtained.

Ref. [16] by Ju and Niu introduces an approximate non-linear ordinary differential equation for the displacement due to a sudden external pressure of an incompressible spherical membrane made of hyperelastic material modeled with the Ogden law. Below a critical pressure for which structural failure is observed, the dynamic response of the membrane is a non-linear periodic oscillation. In the same spirit, the work of Ren [17] determines the critical suddenly exerted pressure for an incompressible neo-Hookean tube. Below the critical pressure, the fixed point is a center surrounded by an homoclinic orbit and the shell undergoes non-linear periodic oscillations. However, above this threshold, the fixed point becomes an unstable saddle node, the phase-plane portrait features a non-closed curve and the shell's displacements ultimately diverge.

Haslach and Humphrey [18] studied the transient vibrations of an isotropic incompressible hyperelastic spherical membrane in a fluid under initial perturbation. The membrane is assumed to deform uniformly and keeps a spherical shape for all times. A single non-linear ordinary differential equation driving the dynamics of this membrane is solved. A few material models, such as the Fung and Skalak biological model as well as the neo-Hookean, Mooney–Rivlin, Alexander and Ogden rubber models are considered in order to clarify which one is the most suitable for the description of the dynamic behavior of soft biological tissues. It is shown that rubber structures exhibit bifurcations in their dynamic response that are not present in traditional models for biological tissues.

In works of Zhu et al., the oscillations of a membrane [19] and a balloon [20] made of dielectric elastomer are studied. The non-linearly elastic behavior of the material is derived from an incompressible neo-Hookean strain energy density with an additional term corresponding to the dielectric energy. In [19], the superharmonic, harmonic, and subharmonic resonances, caused by an external sinusoidal pressure are found in agreement with experimental data previously obtained by Fox and Goulbourne [21,22]. In [20], the stability of the static equilibrium is investigated along with the surrounding oscillations. It is found that the vibration amplitude of the balloon may jump at certain values of the excitation frequency, exhibiting hysteresis. However, only a low-dimensional model is used in this study. Complementary results on the dynamics of a dielectric incompressible neo-Hookean spherical shell are reported in [23]: the system is more stable with an increasing thickness of the shell.

Yuan et al. [24] studied the dynamic inflation of infinitely long cylindrical tubes. The inner surface of the tube is subjected to periodic step pressures. The incompressible Ogden material model is implemented and the strain energy density function reduces to neo-Hookean type. Sensitivity of the dynamics to the material parameters, the structure geometry, and the applied pressure is discussed. In [25], the authors explored the dynamic symmetric response of a spherical membrane made of incompressible material with the Rivlin–Saunders constitutive law. The membrane is subjected to periodic step loads on its inner and outer surfaces. A spherically symmetric deformation is assumed and the expressions for principal stretches are derived. The corresponding differential equation that approximately drives the normal oscillation of the spherical membrane is obtained. It is found that periodic oscillations do not exist and that the shell will be fractured ultimately with time when the prescribed external load exceeds a critical threshold that depends on material parameters.

Ogden and Roxburgh studied plane vibrations superimposed to the pure homogeneous deformation of a rectangular plate made of incompressible [26] and compressible [27] hyperelastic materials. The sensitivity of the frequency spectrum to the pre-stress and geometrical parameters for a plane-strain plate problem is provided for neo-Hookean, Varga and Blatz–Ko hyperelastic laws.

Gonçalves et al. [9,28] investigated linear and non-linear free and forced vibrations of pre-stretched annular hyperelastic membranes. The membrane material is assumed to be incompressible, homogeneous, isotropic, and is described by a neo-Hookean constitutive law. The in-plane displacements are ignored while the transverse displacement field is approximated by a series of natural modes. Both studies report that a single degree-of-freedom model correctly predicts large vibrations. It is also shown that (i) a lightly stretched membrane displays a highly non-linear hardening response, (ii) the non-linearity decreases as the stretching ratio increases, and (iii) the response becomes essentially linear for a deformed radius of at least twice the initial value. In Ref. [9], a comparison is conducted for different hyperelastic models such as the Mooney–Rivlin, Yeoh, Ogden and Arruda–Boyce models. The results show that the membrane exhibits the same non-linear frequency–amplitude behavior for all tested models, with just a slight difference for the Ogden model.

This literature review highlights the fact that previous works, in a vast majority, employ incompressible neo-Hookean laws to describe rubbers' non-linear elastic behaviors: accordingly, this model is also considered in the present investigation, which extends hyperelastic plate finite amplitude bending vibrations to commonly ignored in-plane displacements. Also, the common and simplifying pre-assumed shape of deformation is discarded here since potential internal resonances can combine several modes in the sought vibratory signatures, making the targeted deformation pattern quite complicated [4]. The present study attempts to overcome the above-mentioned restriction for the large-amplitude static as well as vibratory investigations of a rubber rectangular plate. The von Kármán model is exploited to capture geometrically non-linear bending as it is reputed to be sufficiently accurate for thin isotropic plates [4]. The associated governing equations neglect shear strains and rotary inertia; in addition, thickness stretching/compression cannot be readily recovered and the transverse normal strain has to be determined through additional considerations. Accordingly, the absence of transverse normal stress together with an incompressibility constraint is utilized within a method that iteratively builds a local model capable of accurately approximating the plate's vibratory response around a static deformed configuration. A comparison with a purely geometrically non-linear model is carried out.

2. Lagrange equations

The static and dynamic deflections of a thin rubber plate are considered and the well-known Lagrange framework is used to derive the corresponding governing equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=Q_i, \tag{1}$$

where $L=T-\Pi$ is the Lagrange's functional of the system of interest; it comprises the potential energy of elastic deformation $\Pi=\iiint_V W\,\mathrm{d}v$, integral of the strain energy density W over the volume V, and the kinetic energy of the plate, T expressed as [4]

$$T=\frac{1}{2}\rho h\int\int_S(\dot{u}^2+\dot{v}^2+\dot{w}^2)\,\mathrm{d}s, \tag{2}$$

where S is the surface area of the middle plane of the plate; ρ is the pass-density per unit length of the plate material, and h is the thickness of the plate. In Eq. (1), the displacements $u(x,y,t)$,