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On the mathematical paradoxes for the flow of a viscoplastic film down an inclined surface

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ABSTRACT

In this paper we consider the motion of thin visco-plastic Bingham layer over an inclined surface whose profile is not flat. We assume that the ratio between the thickness and the length of the layer is small, so that the lubrication approach is suitable. Under specific hypotheses (e.g. creeping flow) we analyze two cases: finite tilt angle and small tilt angle. In both cases we prove that the physical model generates two mathematical problems which do not admit non-trivial solutions. We show that, though the relevant physical quantities (e.g. stress, velocity, shear rate, etc.) are well defined and bounded, the mathematical problem is inherently ill posed. In particular, exploiting a limit procedure in which the Bingham model is retrieved from a linear bi-viscous model we eventually prove that the underlying reason of the inconsistency has to be sought in the hypothesis of perfect stiffness of the unyielded part. We therefore conclude that: either the Bingham model is inappropriate to describe the lubrication motion over a non-flat surface, or the lubrication technique fails in approximating thin Bingham films.

1. Introduction

Several authors have studied visco-plastic or Bingham flows [35,7,22,37,6,12,28,36,24] and in particular the so-called " lubrication paradox" for yield stress fluids [28,22]. The specific case of a visco-plastic " thin film" flowing down an inclined surface has been widely investigated [23,1,27,34] and we refer the reader to the recent papers [3,21] for a general overview.

In this paper we study the flow of a Bingham fluid down a non-flat surface[1] $y^*=b^*(x^*)$ under the sole action of gravity (see Fig. 1). Referring to Fig. 1, we set $\alpha=\arcsin(D^*/L^*)$ as the tilt angle and we assume that db^*/dx^* is "small" but not zero (non-flat surface).

We assume that the ratio ε between the thickness of the fluid layer and the length of the surface is sufficiently small (lubrication approximation) and we expand the physical quantities in powers of ε, matching the corresponding terms in the governing equations. We focus on a fully developed flow, not considering the thorny issue of the advancing wetting front. We analyze the following cases[2]:

(**1**) $\tan\alpha=\mathcal{O}(1)$ (Section 3);
(**2**) $\tan\alpha=\mathcal{O}(\varepsilon)$ (Section 4);

We show that, unless $b^*(x^*)$ does not have a specific profile (polynomial of order $n\le 2$ for case (**1**), or $b^*\equiv 0$, for case (**2**)), the classical Bingham model gives rise to mathematical paradoxes like the ones encountered in the channel flow, [22]. We remark that compatibility issues between top and bottom boundary conditions arise also when both contact friction and Coulomb friction are considered (see, e.g. [26]). In particular, we show that the inconsistency lies not on the divergence of the stress at the "solid-fluid" interface, but is intrinsically related to the mathematical structure of the problem, which does not admit solutions that fulfill the basic physical requirements. As a consequence we conclude that: either the Bingham model may be inadequate to describe the "lubrication" flow over a generic surface, or the "lubrication" technique fails in approximating the thin film flow of a Bingham fluid.

The lubrication paradox is an "old problem" (see again [22]) and some of the authors that have investigated it (see, e.g. [1]) claim that the paradox is only apparent. The main motivation for this claiming is that the asymptotic expansion seems to break down at the yielding interface as it seems to give rise to unbounded diagonal stress components. Accordingly the paradox must be ascribed to an improper scaling which has to be corrected in order to avoid diverging stress. Our opinion disagree with this conclusion as we have found that the stress is actually bounded on the yield surface. In particular we prove that:

(a) The stress components are all bounded at both ε^0, and ε^1 orders (and so the scaling is correct).
(b) Velocities and interfaces are well defined and bounded at the ε^0 order.
(c) The paradox arising from the model is purely mathematical. Indeed we do not find any physical inconsistency (e.g.

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[1] Throughout this paper the superscript " * " denotes dimensional variables.
[2] With $\mathcal{O}(1)$ we mean a quantity " well separated" from 0 and finite.

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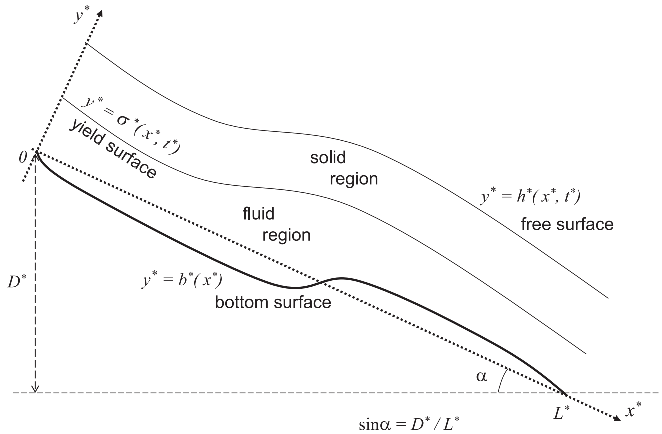

Fig. 1. Sketch of the flow (not on scale).

unbounded stress components), but we show that the mathematical structures of the problems are inherently ill posed. We actually prove that the BVP's do not admit solutions that are compatible with the constraints imposed by the kinematics. Actually, it would be more correct to say that the paradox stems from an incorrect modeling approximation (but we shall return on this issue at the end of the introduction).

(d) The mathematical inconsistencies may also be highlighted considering a "bi-viscous" model which tends to the classical Bingham. This issue is illustrated in some detail in Appendix A.

The paper develops as follows: in Section 2 we illustrate the basic assumptions and we give a necessary condition for the solutions to be physically meaningful (Proposition 1). Then we develop the model assuming that the solutions are consistent with Proposition 1 and we check whether the mathematical problems admit solutions or not.

In Section 3 we introduce the equations approximated at the zero order for case (**1**) and we illustrate (Proposition 2) the kinematic constraint imposed by Proposition 1. In particular, in Appendix A, exploiting entropy arguments, we show that the "bi-viscous" model tends to the one of Bingham only if the constraint imposed by Proposition 2 is fulfilled.

Section 3.2 is devoted to the mathematical contradiction. Here we prove (Theorem 2) that the mathematical problem is uniquely solvable (in the sense of Propositions 1 and 2), if two conditions are fulfilled: (*i*) $b^*(x^*)$ is flat or is a parabola, and (*ii*) the inlet discharge is constant in time. In all the other cases (e.g. bottom surface flat but inlet discharge varying in time) the mathematical problem is simply ill posed. In Section 4, proceeding in an analogously way, we analyze case (**2**), proving that the mathematical problem is well posed only if $b^*(x^*)$ is flat and the inlet discharge is constant (Theorem 3).

Finally in Section 5 we present two tables listing the various results, which provides a summary of the solvability/non-solvability of the mathematical problems.

The achievement of the above mentioned results was facilitated of having used a fully implicit constitutive Bingham-like model, as recently stressed by Rajagopal [29–32]. The advantage of using such an approach relies mainly in the fact that the asymptotic expansions can be easily treated. Indeed the matching between the corresponding terms is straightforward (see Remark 6). We refer the readers to the paper [33] for an interesting application of lubrication approximation to a fluid whose constitutive equation is defined implicitly.

A possible way for overcoming the mathematical paradoxes that arise in the lubrication approximation might be to consider a model allowing for deformations of the "unyielded phase". In this sense we refer to the pioneering work of Oldroyd [25], Yoshimura and Prud'homme [38] and to the recent works by Fusi and Farina [8–11] and [13–18]. Of course, it is quite possible that the full system of equations generated by the Bingham model do not exhibit paradoxical behavior. Indeed, the paradoxes we encounter here are closely tied to the peculiar geometrical setting that we are considering. However, if this were true, we simply conclude that the asymptotic technique is not applicable to the thin Bingham flows.

2. Geometrical setting, kinematics and governing equations

We consider a continuum, modeled as an incompressible Bingham fluid, that flows down an inclined surface forming a layer of varying width, as schematically depicted in Fig. 1. The discharge per unit layer width, $Q^*(t^*)$, is prescribed. In particular, we set

$$Q^*(t^*) = q(t^*) Q^*_{ref}, \quad [Q^*_{ref}] = \mathrm{m}^2/\mathrm{s}, \tag{2.1}$$

where $q = \mathcal{O}(1)$ and Q^*_{ref} represent the order of magnitude of the discharge. Denoting by ρ^* the material density (uniform and constant), by η^* the viscosity and by H^* the characteristic thickness of the layer we find (see e.g. [2])

$$H^* = \sqrt[3]{\frac{\eta^* Q^*_{ref}}{\rho^* g^* \sin\alpha}}, \tag{2.2}$$

where g^* is gravity acceleration. Denoting by L^* the longitudinal length scale, we introduce the parameter

$$\varepsilon = \frac{H^*}{L^*}, \tag{2.3}$$

and rescale the spatial variables as

$$x = \frac{x^*}{L^*}, \quad y = \frac{1}{\varepsilon}\frac{y^*}{L^*}. \tag{2.4}$$

As mentioned in the introduction we assume

$$\varepsilon \ll 1, \tag{2.5}$$

so that the "lubrication approximation" is suitable.

We introduce the Eulerian velocity

$$\mathbf{u}^*(\mathbf{x}^*, t^*) = u_1^* \mathbf{e}_x + u_2^* \mathbf{e}_y,$$

and we rescale the bottom surface as $b^* = bB^*$, where $B^* = \max_{x^* \in [0,L^*]} |b^*(x^*)|$. Since we do not want to deal with detachment phenomena, we consider

$$\max_{x^* \in [0,L^*]} \left| \frac{db^*}{dx^*} \right| = B \max_{x \in [0,1]} \left| \frac{db}{dx} \right| \ll 1. \tag{2.6}$$

where $B = B^*/L^*$. Indeed, to be more precise, we assume $B = \mathcal{O}(\varepsilon)$, and $\max_{x \in [0,1]} |db/dx| = \mathcal{O}(1)$, and, to keep notation simple, we set $B = \varepsilon$ (see also Remark 2). Next, introducing the unit normal and unit tangent vectors to the bottom surface $\mathbf{n}_b$, and $\mathbf{t}_b$, respectively, we define

$$u_n^* = \mathbf{u}^*(x^*, b^*(x^*)) \cdot \mathbf{n}_b = \frac{1}{\sqrt{1 + \left(\varepsilon \dfrac{db}{dx}\right)^2}} \left(-\varepsilon \frac{db}{dx} u_1^* + u_2^* \right), \tag{2.7}$$

$$u_t^* = \mathbf{u}^*(x^*, b^*(x^*)) \cdot \mathbf{t}_b = \frac{1}{\sqrt{1 + \left(\varepsilon \dfrac{db}{dx}\right)^2}} \left(u_1^* + u_2^* \varepsilon \frac{db}{dx} \right). \tag{2.8}$$

Finally we denote by U^* the characteristic longitudinal velocity of the fluid that, recalling (2.2), yields

$$U^* = \frac{Q^*_{ref}}{H^*} = \sqrt[3]{\frac{\rho^* g^* Q^{*2}_{ref} \sin\alpha}{\eta^*}} = \frac{\rho^* g^* H^{*2}}{\eta^*} \sin\alpha. \tag{2.9}$$

Remark 1. A more general approach to the problem should consider a set of curvilinear coordinates $(\varkappa^*, \xi^*)$ whose unit vectors