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Time-delayed stochastic optimal control of strongly non-linear systems with actuator saturation by using stochastic maximum principle

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ABSTRACT

A time-delayed stochastic optimal bounded control strategy for strongly non-linear systems under wide-band random excitations with actuator saturation is proposed based on the stochastic averaging method and the stochastic maximum principle. First, the partially averaged Itô equation for the system amplitude is derived by using the stochastic averaging method for strongly non-linear systems. The time-delayed feedback control force is approximated by a control force without time delay based on the periodically random behavior of the displacement and velocity of the system. The partially averaged Itô equation for the system energy is derived from that for the system amplitude by using Itô formula and the relation between system amplitude and system energy. Then, the adjoint equation and maximum condition of the partially averaged control problem are derived based on the stochastic maximum principle. The saturated optimal control force is determined from maximum condition and solving the forward–backward stochastic differential equations (FBSDEs). For infinite time-interval ergodic control, the adjoint variable is stationary process and the FBSDE is reduced to a ordinary differential equation. Finally, the stationary probability density of the Hamiltonian and other response statistics of optimally controlled system are obtained from solving the Fokker–Plank–Kolmogorov (FPK) equation associated with the fully averaged Itô equation of the controlled system. For comparison, the optimal control forces obtained from the time-delayed bang–bang control and the control without considering time delay are also presented. An example is worked out to illustrate the proposed procedure and its advantages.

1. Introduction

Optimal control of non-linear stochastic dynamic systems is an important research area due to its relevance to many engineering applications. The most widely used tool in solving stochastic optimal control problem is stochastic dynamic programming [1,2]. However, so far only linear quadratic Gaussian (LQG) strategy has been widely used in engineering, even for non-linear stochastic systems [3], due to the difficulty in solving high-dimensional non-linear HJB equation. In the last decade, a non-linear stochastic optimal control strategy for non-linear quasi-Hamiltonian (and generalized Hamiltonian) systems has been proposed by Zhu and his co-workers [4] based on the stochastic averaging method for quasi-Hamiltonian systems and dynamic programming principle. Later, this control strategy was extended to stabilization [5] and reliability maximization [6] of quasi-Hamiltonian systems. In practical control problem, system uncertainty, time delay and actuator saturation, etc. have to be considered. Thus, the techniques for stochastic optimal control of partially observable systems [7], stochastic optimal time-delay control [8], stochastic optimal semi-active control [9] and stochastic mini-max control [10] have also been developed.

The maximum principle is widely used in the optimal control of deterministic systems. However, in solving stochastic optimal control problems, the stochastic maximum principle has been less applied due to the difficulty in solving the resultant FBSDE, although some methods for solving this equation have been developed [11–13]. An optimal control strategy combining the stochastic averaging method and the stochastic maximum principle has been proposed by the authors of the present paper through making some reasonable assumptions [14]. However, several problems have to be solved before it is applied to engineering systems. The time delay is usually unavoidable due to the time spent in measuring and estimating the system state, calculating and executing the control forces. The effects of time delay have been studied for different systems [15–17]. It is shown that without considering the effect of time delay, the control force may even destabilize the system [18]. The other practical problem is that the control force should be bounded due to saturation of

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actuator. The saturated linear control of linear systems has been developed [19,20]. The saturated control strategy for quasi-Hamiltonian systems has been proposed by Zhu and his co-workers [21–23].

Strongly non-linear systems under wide-band random excitations often occur in engineering. In the present paper, a time-delayed optimal control strategy for strongly non-linear systems under wide-band random excitations with actuator saturation is proposed based on stochastic averaging and stochastic maximum principle. By using the stochastic averaging method, the original control problem is converted to that for system energy, and the resultant FBSDE is only two dimensional. The time-delayed feedback control force is approximated by a control force without time delay due to the randomly periodic behavior of the displacement and velocity of the system. The optimal bounded control force is obtained through the maximum condition derived from the stochastic maximum principle. An example is worked out to illustrate the effectiveness and efficiency of the proposed control strategy.

2. Formulation of original optimal control problem

Consider the following controlled strongly non-linear oscillator subject to lightly linear and (or) non-linear damping and weak external and (or) parametric excitations of wide-band noise:

$$\ddot{X}+g(X)=\varepsilon h(X,\dot{X})+\varepsilon^{1/2}f_k(X,\dot{X})\xi_k(t)+\varepsilon u_\tau$$
$$k=1,2,\ldots,m,\quad |u_\tau|\le b_0 \tag{1}$$

where $g(X)$ represents a strongly non-linear restoring force; ε is a small parameter; εh represents lightly linear and/or non-linear damping force; $\varepsilon^{1/2}f_k$ denote the amplitudes of random excitations; $u_\tau=u(X_\tau,\dot{X}_\tau)$ is the time-delayed feedback control with constraint $|u_\tau|\le b_0$; $\xi_k(t)$ are wide-band stationary ergodic random processes with zero mean and correlation functions $R_{kl}(\tau)$ or spectral densities $S_{kl}(\omega)$.

The objective of present study is to determine a time-delayed bounded control law to minimize the response of the system (1), which is expressed in terms of the following performance index:

$$J(u_\tau(\cdot))=E\left[\int_0^T L(X,\dot{X},u_\tau)\,dt+\psi(X(T),\dot{X}(T))\right]$$
$$|u_\tau|\le b_0 \tag{2}$$

for finite time-interval control, or

$$J(u_\tau(\cdot))=\lim_{T\to\infty}\frac{1}{T}\int_0^T L(X,\dot{X},u_\tau)\,dt$$
$$|u_\tau|\le b_0 \tag{3}$$

for infinite time-interval ergodic control. In Eqs. (2) and (3), $L(X,\dot{X},u_\tau)$ is the cost function, T is terminal control time, $\psi(X(T),\dot{X}(T))$ is the terminal cost. Eqs. (1), (2), or (1), (3) constitute the mathematical formulation of the original optimal control problem.

3. Converted optimal control problem

The unavoidable time delay of control force will cause great difficulties in optimal control design. In this section, two steps are adopted to simplify the control problem. First, the original time-delayed control force will be approximated by the control force without time delay based on the randomly periodic behavior of the system states. Then, the original control problem is converted to that of the system energy by using the stochastic averaging method.

3.1. Approximate control force without time delay

Assume that when $\varepsilon=0$, system (1) has a family of periodic solutions. Then, when ε is small, the response of system (1) will be randomly periodic and can be written as

$$X(t)=A\cos\Psi(t)+B$$
$$\dot{X}(t)=-Av(A,\Psi)\sin\Psi(t) \tag{4}$$

where

$$\Psi(t)=\Phi(t)+\Theta(t)$$
$$v(A,\Psi)=\frac{d\Phi}{dt}=\sqrt{\frac{2[U(A+B)-U(A\cos\Psi+B)]}{A^2\sin^2\Psi}} \tag{5}$$

in Eqs. (4) and (5), A,B,Ψ,Φ and Θ are random processes, $U(X)=\int_0^X g(x)\,dx$ is the potential energy.

Under the assumption that the time delay τ is small, the following approximate relations hold [8]:

$$X_\tau=X(t-\tau)\approx X\cos\omega\tau-\frac{\dot{X}}{\omega}\sin\omega\tau$$
$$\dot{X}_\tau=\dot{X}(t-\tau)\approx\dot{X}\cos\omega\tau+X\omega\sin\omega\tau \tag{6}$$

Thus, the time-delayed feedback control force $u(X_\tau,\dot{X}_\tau)$ in Eq. (1) can be replaced by the control forces without time delay, i.e., $u(X_\tau,\dot{X}_\tau)\approx u(X,\dot{X};\tau)$.

3.2. Stochastic averaging

Treating Eq. (4) as generalized van der Pol transformations from X, $\dot{X}$ to A,Ψ. The following equations for A and Ψ can be derived from Eq. (1):

$$\frac{dA}{dt}=\varepsilon m_1(A,\Psi)+\varepsilon m_1^u(A,\Psi,u(A,\Psi;\tau))+\varepsilon^{1/2}\sigma_{1k}\xi_k(t)$$
$$\frac{d\Psi}{dt}=\omega(A)+\varepsilon m_2(A,\Psi)+\varepsilon m_2^u(A,\Psi,u(A,\Psi;\tau))+\varepsilon^{1/2}\sigma_{2k}\xi_k(t) \tag{7}$$

where $\omega(A)$ is the average frequency and

$$m_1=\frac{-A}{g(A+B)(1+h)}h(A\cos\Psi+B,-Av(A,\Psi)\sin\Psi)v(A,\Psi)\sin\Psi$$
$$m_1^u=\frac{-A}{g(A+B)(1+h)}u(A,\Psi;\tau)v(A,\Psi)\sin\Psi$$
$$m_2=\frac{-1}{g(A+B)(1+h)}h(A\cos\Psi+B,-Av(A,\Psi)\sin\Psi)v(A,\Psi)\cos\Psi$$
$$m_2^u=\frac{-1}{g(A+B)(1+h)}u(A,\Psi;\tau)v(A,\Psi)\cos\Psi$$
$$\sigma_{1k}=\frac{-A}{g(A+B)(1+h)}f_k(A\cos\Psi+B,-Av(A,\Psi)\sin\Psi)v(A,\Psi)\sin\Psi$$
$$\sigma_{2k}=\frac{-1}{g(A+B)(1+h)}f_k(A\cos\Psi+B,-Av(A,\Psi)\sin\Psi)v(A,\Psi)\cos\Psi \tag{8}$$

$$h=\frac{g(-A+B)+g(A+B)}{g(-A+B)-g(A+B)} \tag{9}$$

It is seen from Eq. (7) that since ε is small so A is slowly varying process while Ψ is rapidly varying process. Based on the stochastic averaging method for strongly non-linear systems [24], $A(t)$ converges weakly to a diffusive Markov process as $\varepsilon\to0$ and is governed by the following averaged Itô equation:

$$dA=[m(A)+\varepsilon\langle m_1^u\rangle_t]\,dt+\sigma(A)\,dB(t) \tag{10}$$

where

$$m(A)=\varepsilon\left\langle m_1+\int_{-\infty}^0\left[\left(\frac{\partial\sigma_{1k}}{\partial A}\Big|_t\sigma_{1l}|_{t+\tau}+\frac{\partial\sigma_{1k}}{\partial\Theta}\Big|_t\sigma_{2l}|_{t+\tau}\right)R_{kl}(\tau)\right]d\tau\right\rangle_t$$