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Two-stage optimal stochastic linearization in analyzing of non-linear stochastic dynamic systems ☆

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ABSTRACT

A novel stochastic linearization approach is developed to predict the second-moment response of non-linear systems under stochastic parametric and external excitations. The present approach is realized by a two-stage optimization: the first stage of optimal linearization modeling and the second stage of parameters optimization. Five examples, including two polynomial oscillators, one hysteretic Bouc-Wen oscillator under stochastic external excitation, and two polynomial oscillators under stochastic parametric and external excitations are selected to illustrate the present approach. The validity of the present approach is validated by some approximate solutions, exact solutions, and Monte Carlo simulations. For system non-linearity, which can be approximated as a full-states linear combination in the Gaussian linearization model, the present approach offers a more accurate prediction of the second moment than that by the Gaussian linearization method. The two-stage optimal Gaussian linearization method incorporates the merits of Gaussian linearization method in the first stage and the SPEC-alternative in the second stage.

1. Introduction

The investigation of the dynamic behavior of stochastic non-linear systems has attracted numerous researchers in the areas of mechanical, structural, and control engineering for over fifty years [1,2]. For engineering stochastic non-linear problems, the investigation of dynamic response of the first two moments is the most essential; however, the complete dynamic behavior can only be derived from the probability density response. By modeling an engineering stochastic non-linear system as an Ito's equation, in principle, the associated density response can be obtained by solving the Fokker–Planck–Kolmogorov (FPK) equation [3]. In practice, even for stationary solution, the exact density function can be derived only for certain specific classes of stochastic externally excited non-linear systems, not to mention with stochastic parametric excitation, where more challenged issues need to be considered [3]. For this reason, it strongly depends on approximate methods and numerical methods for finding solutions. There are several approximate solution methods including Gaussian linearization method, cumulant-neglect closure method, perturbation method, Gram-Charlier expansion method, equivalent external excitation method, maximum entropy method, information closure method, approximate methods of solving FPK equation, etc., which have developed and extended to predict the statistical responses of non-linear systems [3–11]. Recent new advances of path integral [12–14] and probability density evolution methods [15,16] were proposed to obtain approximate non-stationary probability density responses. Among those solution schemes, Gaussian linearization method has been the most-employed for the purpose of analyzing the first two moments response of general non-linear stochastic systems in engineering applications [4–7].

Historically, the earliest work of statistical linearization was developed independently by Booton (1953) and Kazakov (1954), and equivalent linearization was contributed by Caughey (1959) [4–7]. Although there are some subtle differences between them [4,5,17], stochastic linearization, statistical linearization, equivalent linearization, or stochastic equivalent linearization have all been used in agreement as the standard method to analyze statistical responses of non-linear dynamic systems under stochastic external excitations [4–7,17]. In the standard method, Gaussian density is utilized in the mean-square linearization scheme. The minimization of mean-square linearization error was extended to that of energy deviation by Wang and Zhang [18], Zhang et al. [19], and Elishakoff and Zhang [20] for better predicting the second moment. The scheme of mean-square energy deviation also showed improved accuracy of the second-moment response for Duffing oscillator under colored noise excitations [21]. Socha and Pawleta

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(1994), and later by Elishakoff and Colojani (1997), proposed a linearization method with an attempt to correct the inconsistent derivation of linearization coefficients in the standard linearization method [5]. The method, later recognized as a consistent realization of notice in Lin's book [2,7], is named as SPEC-alternative by Crandall [17]. Actually, the name of SPEC-alternative is more precisely as SPEC-alternative of Elishakoff and Colojani (SPEC-alternative-EC).

In recognizing the non-Gaussian response of a general non-linear stochastic system, a non-Gaussian density is employed in the mean-square linearization for better prediction of the second-moment responses [5,17]. It was found that if a non-Gaussian density, which has the same shape of true distribution, is employed to replace the Gaussian density in the standard linearization method, the exact second-moment response can be derived. The property of non-Gaussian density in linearization was initially stated and proved by Caughey (1963), and later by Crandall (1979), and named as true linearization by Kozin (1988) [7]. For the linearization method of applying non-Gaussian density to evaluate the linearization coefficients, the linearization approach is named as non-Gaussian linearization [22].

The standard linearization method for systems under external Gaussian white noise excitation has been extended to include parametric Gaussian white noise excitation [5,8,22–26]. For non-linear systems under parametric Poisson white noise excitation, the validity of utilizing different Gaussian linearization schemes in the second-moment response was compared [27]. Methods of Gaussian linearization with parametric white noise have been further extended to non-Gaussian linearization for better prediction of statistical responses [22,23,25].

The number of publications on stochastic linearization is very large. A complete survey on the related literatures is almost impossible. For other attempts to extend and improve the stochastic linearization approach, a survey up to 2007 had been given in the book by Socha [5]. Recently, wavelets-based equivalent linearization methods which follow the concept of stochastic linearization were developed to predict time-domain and frequency-domain non-stationary stochastic response [28,29].

In the prediction of the mean and second-moment responses, Gaussian linearization method takes advantage of easy implementation, computational efficiency, acceptable accuracy, and mostly, applicability to versatile non-linear high-dimensional engineering systems. For zero-mean response, the accuracy in predicting the second moment, in general, decreases as the strength of non-linearity increases. For non-linear systems subjected to stochastic external excitations, the accuracy in predicting the second moment by the SPEC-alternative-EC is less than that of the standard Gaussian linearization method. However, the SPEC-alternative-EC provides more accurate optimal linearization coefficients than those of the standard Gaussian linearization method [17]. The advantages and disadvantages of SPEC-alternative compared with the standard method were argued and debated in literatures [5,17]. The subtle differences in the statistical linearization method, equivalent linearization method, and SPEC-alternative are easily ignored and misused in the formulation of Gaussian linearization. For the clarification of the differences in formulation, causal block diagrams were defined and proposed for the representation of three linearization methods [30]. From the block-diagram representations, the differences in linearization modeling and evaluation of the second moment of three linearization methods were clearly identified. In addition, an improved Gaussian linearization method to incorporate the merits of the standard method and SPEC-alternative was proposed.

In this paper, a novel stochastic linearization approach for the prediction of the second-moment response is formulated for general stochastic non-linear systems. Five examples of non-linear stochastic oscillators are selected to elucidate the applications of the proposed method. Finally, the application and performance of the present approach are concluded.

2. Formulation of two-stage optimal stochastic linearization

A novel stochastic linearization approach for stochastic parametrically and externally excited non-linear systems is proposed. The proposed stochastic linearization approach is an extension of the improvement method, which is formulated for stochastic externally excited non-linear systems [30].

Consider a general n-dimensional, non-linear stochastic system described as

$$dX(t) = F(X(t))dt + G(X(t))dW(t)$$
$$X(t_0) = X(0), \tag{1a}$$

where $X(t)$ is an $n \times 1$ vector of state processes and $X(0)$ is an initial condition with given distribution. $F(X(t))$ is an $n \times 1$ vector of linear and non-linear function of states, $G(X(t))$ is an $n \times m$ matrix of linear and non-linear function of states, and $F(X(t))$ and $G(X(t))$ satisfy the Lispschitz and growth conditions for the existence of mean-square stationary solution. $W(t)$ represents a zero-mean $m \times 1$ vector Wiener process with intensity

$$E[dW(t)dW^T(t)] = Q_w dt. \tag{1b}$$

The presented approach consists of two major stages. The first stage is to derive a structure of linearization model by minimizing the mean-square error between a system and its model [25]. On the other hand, the second stage is to optimize the parameters of the linearization model for the minimization of modeling error. For the first stage, the linearization model of the system (1a) can be described as

$$dX(t) = H_1(X(t))dt + H_2(X(t))dW(t)$$
$$X(t_0) = X(0), \tag{2}$$

where $H_1(X(t))$ and $H_2(X(t))$ are linear functions of states $X(t)$. For the linearization model in (2), the model includes linear state-noise multiplicative terms. Thus, the linearization model is not a linear but a bilinear form. The approximation of non-linear matrix functions $F(X(t))$ and $G(X(t))$ by $H_1(X(t))$ and $H_2(X(t))$, respectively, is written as

$$F(X(t)) \approx H_1(X(t))$$
$$G_i(X(t)) \approx H_{2,i}(X(t)) \tag{3a}$$

$$H_1(X(t)) = C + A(X(t) - M(t))$$
$$H_{2,i}(X(t)) = B_i + L_i(X(t) - M(t)), \tag{3b}$$

where $G_i(X(t))$, $H_{2,i}(X(t))$, with $i = 1 \sim m$, is the ith partitioned column vector of $G(X(t))$, $H_2(X(t))$, respectively, and $M(t)$ is the mean vector of states $X(t)$ as

$$M(t) = E[X(t)] \tag{4}$$

The approximation errors in (3) for given $F(X(t))$ and $G_i(X(t))$ are

$$\varepsilon_1 = F(X(t)) - C - A(X(t) - M(t)) \tag{5a}$$

$$\varepsilon_{2,i} = G_i(X(t)) - B_i - L_i(X(t) - M(t)). \tag{5b}$$

For minimizing the mean square error of (5a) and (5b), one proceeds with

$$\frac{\partial E[\varepsilon_1^T \varepsilon_1]}{\partial C} = 0, \quad \frac{\partial E[\varepsilon_1^T \varepsilon_1]}{\partial A} = 0, \quad \frac{\partial E[\varepsilon_{2,i}^T \varepsilon_{2,i}]}{\partial B_i} = 0, \quad \frac{\partial E[\varepsilon_{2,i}^T \varepsilon_{2,i}]}{\partial L_i} = 0. \tag{6}$$

By assuming that the operations of expected values in (6) are independent of the linearization coefficients C, A, B_i, and L_i, the linearization matrix C, A, B_i, and L_i in (3b) are derived,