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Analysis of laser power threshold for self oscillation in thermo-optically excited doubly supported MEMS beams

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ABSTRACT

An optically thin MEMS beam suspended above a substrate and illuminated with a CW laser forms an interferometer, coupling out-of-plane deflection of the beam to absorption within it. In turn, laser absorption creates thermal stresses which drive further deflection. This coupling of motion to thermal stresses can cause limit cycle oscillations in which the beam vibrates in the absence of periodic external forcing. Prior work has modeled such thermal–mechanical systems using ad-hoc coupled ordinary differential equations, with finite element analysis (FEA) used to fit model parameters. In this paper we derive a first principles model of such oscillations from the continuum description of the temperature and displacement field. A bifurcation analysis of the model is performed, allowing us to easily estimate the threshold power for self-oscillation as a function of geometric and optical constants of the beam.

1. Introduction

Due to their high frequency, microelectromechanical system (MEMS) resonators have been proposed for a number of different sensing [1] and signal processing [2–4] applications in the past decade. Typically, MEMs resonators are capacitively or piezoelectrically driven using a sinusoidal signal. Shifts in their resonant frequency or phase relationship can be used to infer a measurement or perform a calculation. Such drive methods require an external, highly stable frequency source and additional conductive or piezoelectric material layers on the device. Optical excitation methods can produce self-oscillation without the need for an external periodic excitation, or additional device layers.

Previous work [5–7] has shown that an optically thin MEMS device suspended over a substrate sets up a Fabry–Pérot interferometer which couples absorption and deflection. Illuminating the device with a continuous wave (CW) unmodulated laser causes optical–thermal–mechanical feedback. For low laser power, the device will bend statically, but for high enough laser power it has been observed experimentally [8] that such devices may undergo a Hopf-bifurcation leading to self-oscillation. Similar phenomena include thermal–mechanical feedback oscillations in satellites subjected to solar radiation [9], and aero-elastic feedback oscillations (flutter) in aircraft [10].

For interferometric transduction of MEMS resonators to be a viable means of producing periodic motion, first its causes must be understood, and then models developed that predict the minimum laser power needed for self-oscillation. Several researchers have given explanations of the causes of self-oscillation. Churenkov [6] examined beams with surface coatings and showed that differing coefficients of thermal expansion between the beam material and surface coating could cause bending moments that drive oscillation. Langdon and Dowe [5] assumed that energy was absorbed near the top surface and that vertical thermal gradients caused bending moments which drove self-oscillation. Both use energy methods to derive formulae for the minimum laser power needed to sustain oscillation. However, we have shown [11] that the mechanical–thermal coupling in uncoated pre-buckled beams is due to asymmetry of the anchor support, and not the bi-metallic effect or vertical thermal gradients. Sekaric et al. gives a semi-empirical formula for the threshold power for self-oscillation in [12] based on measured frequency shifts due to heating, though no model of the dynamics. Gigan et al. have suggested that radiation pressure may drive self-oscillation in beams coated for high reflectivity [13–15]. In this work we consider only photo-thermal forces.

Models of device dynamics have also been constructed. Variations of a coupled oscillator model are used to model device dynamics in [8,16–19]. Perturbation theory is used to estimate the threshold power for self-oscillation in [8,16]. Such models provide accurate predictions of the transition power to self-oscillation for specific devices, but would require parametric FEA to study, e.g. the impact of device geometry on the transition power.

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Motivated by the need to understand the contributors to low power self-resonant devices at the level of device design, in this paper we sacrifice accuracy for ease of use and present an (almost) parameter free model of interferometrically driven self-resonant MEMS. Initially straight, doubly supported beams are chosen due to their simple geometry and wide-spread application. Perturbation analysis is used to predict the threshold power for self-oscillation, and predictions compared with the results of numerical continuation. Scalings of threshold power with device geometry and pre-stress are discussed.

2. Mathematical model

Our analysis models doubly supported MEMS beams illuminated with a CW laser focused to a spot at their center. A beam theory model which incorporates in-plane tension is used to model the displacement field of the beam. Device imperfections are an important source of thermal–mechanical coupling in doubly supported beams [11] and are incorporated later in the model. The temperature field is also modeled as a one-dimensional continuum governed by a first order thermal equation. Finally, an optical model determines the laser power absorbed as a function of the beam's center displacement. These two partial differential equations (PDEs) and one algebraic equation describe the optical-thermal–mechanical feedback in the device. A Galerkin projection is used to approximate the PDEs as a set of coupled ordinary differential equations (ODEs), and an imperfection term is added to account for asymmetry of the support. Finally bifurcation analysis of the ODEs is used to estimate the threshold laser power for self-oscillation.

To begin with, we use beam theory to model the mechanical behavior of the beam. Our model is adapted from an equation for the vibration of a beam including membrane stiffness. Only the details are sketched here, and the reader is referred to the original paper [20] for further details. Letting x be the position along the beam, $y(x)$ be the lateral deflection at point x, and including the effects of membrane stress

$$M[y,U] = EIy'''' + Fy'' + m\ddot{y} + \zeta\dot{y} = 0, \tag{1}$$

where m is the mass per unit length, ζ is the viscous damping coefficient, EI is the flexural rigidity, F is the force of tension in the beam, primes denote spatial derivatives, and overdots denote time derivatives. In plane forces arise from residual tension, thermal expansion, and deflection. Using linear thermo-elasticity and writing the axial extension due to deflection to first order, the force of tension, F, is

$$F = \sigma A - \frac{EA}{L}\int_0^L \alpha_e U(x)dx + \frac{EA}{2L}\int_0^L (y'(x))^2 dx, \tag{2}$$

where σ is the residual tensile stress, A is the cross-sectional area of the beam, L is the length, α_e is the coefficient of thermal expansion, and $U(x)$ is the temperature above ambient. The sign convention is that positive loads are tensile and negative loads are compressive. The beam is clamped on both ends giving the boundary conditions

$$y(0) = y(L) = 0, \quad y'(0) = y'(L) = 0. \tag{3}$$

Fourier's law is used to model the temperature field. Since MEMS resonators are often used in low pressure environments to reduce damping, convective heat loss can be ignored. Furthermore, radiative heat loss is negligible for the modest temperature rises predicted. Finally, it has been shown that through thickness thermal gradients are negligible [11]. Thus we model the temperature in the beam using a simple 1D thermal model. Letting $\dot{q}(x)$ be the heat generated per unit volume, and assuming that the temperature above ambient is zero at the ends, we get

$$H[y,U] = \dot{U} - \alpha_c U'' - \frac{1}{\rho c}\dot{q}(x) = 0; \quad U(0) = U(L) = 0. \tag{4}$$

where α_c is the thermal diffusivity, ρ is the mass density, and c is the specific heat capacity. Note that our thermal boundary conditions (4) assume that the substrate acts as an infinite heat sink.

The heating $\dot{q}(x)$ depends on the total laser power P, spatial power distribution, and on the fraction, $f(x)$, of power absorbed at a given distance along the beam. Due to the Fabry–Pérot interferometer between the beam and substrate, $f(x)$ depends on x through the displacement field $y(x)$. Using the optical properties of the films involved, and their thicknesses, $f(y(x))$ can be solved for numerically [21]. However the bifurcation to limit cycle oscillations depends on this function only in the neighborhood of the fixed point, and so we use a Taylor series approximation about zero-deflection.[1] Assuming that the laser power is focused to a spot at the beam's centerline we get

$$\dot{q}(x) = \frac{P}{A}[\alpha_o + \gamma y(x)]\delta\left(x - \frac{L}{2}\right), \tag{5}$$

where P is the total laser power, α_o is the zero-deflection absorption, γ is the contrast of the Fabry–Pérot interferometer, and the delta-function, δ, is our spatial power distribution.

Finally, our equations are projected onto a set of test functions using the Galerkin method, and a set of coupled, non-linear ODEs obtained. For our test functions we select a time-dependent weight function multiplied by a space dependent shape function

$$\tilde{y}(x,t) = a(t)\left(1 - \cos\left(\frac{2\pi x}{L}\right)\right), \quad \tilde{U}(x,t) = b(t)\sin\left(\frac{\pi x}{L}\right). \tag{6}$$

Note that our test functions satisfy the boundary conditions (3) and (4) regardless of the time dependent weight functions. Requiring that our error in approximation be orthogonal to the test functions gives 2 ODEs governing the weight functions

$$\int_0^L \tilde{y}(x,t)M[\tilde{y}(x,t),\tilde{U}(x,t)]dx = 0 \rightarrow \ddot{a} + c_0\dot{a} + c_1 a + c_2 a^3 = c_3 ab, \tag{7}$$

$$\int_0^L \tilde{U}(x,t)H[\tilde{y}(x,t),\tilde{U}(x,t)]dx = 0 \rightarrow \dot{b} = -c_4 b + \frac{2P}{mcL}[\alpha_o + 2\gamma a], \tag{8}$$

with the constants c_i defined as follows:

$$c_0 = \frac{\zeta}{m}$$

$$c_1 = \left[\frac{4\pi^2\sigma A}{3\,mL^2} + \frac{16\pi^4 EI}{3\,mL^4}\right]$$

$$c_2 = \left[\frac{4\pi^4 AE}{3\,mL^4}\right]$$

$$c_3 = \left[\frac{8\pi\alpha_e AE}{3\,mL^2}\right]$$

$$c_4 = \left[\frac{\pi^2\alpha_c}{L^2}\right].$$

Our thermal Eq. (8) is a simple first order thermal equation coupled to the mechanical Eq. (7) through the $2P/mcL[\alpha_o + 2\gamma a]$ term. The mechanical equation is in the form of a damped Duffing oscillator coupled to the thermal equation through the c_{3ab} term. If we neglect the damping and non-linear terms in (7) then our linearized mechanical equation demonstrates the correct frequency relationship inherent in Euler buckling and correctly

[1] This approximation eliminates the periodicity of the interference field and suppresses some phenomenon in the post-Hopf dynamics such as spectral distortion [7,22], limiting amplitude [23], and multi-mode oscillations [23,22]. To first order, it has no impacted on the predicted value of P_{Hopf}.